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Rev. T. E. R. PHILLIPS, PRESIDENT, in the Chair.

John Granville Hawke, Ravensfield, Northumberland Road, New Barnet, Herts ;
Marie Victor Marc Herchenroder, 140 Holland Road, Kensington, London, W. 14 ;
Armin Otto Leuschner, Professor of Astronomy in Berkeley University, California, U.S.A. ; and
Llewellyn Hilleth Thomas, Trinity College, Cambridge,

were balloted for and duly elected Fellows of the Society.

Professor Paul Guthnick, Director of Berlin University Observatory, Neu-Babelsberg, Germany ; and
Professor Michal Kamienski, Director of the University Observatory, Warsaw,

were balloted for and duly elected Associates of the Society.

The following candidates were proposed for election as Fellows of the Society, the names of the proposers from personal knowledge being appended :—

Raffaele Invrea, Royal Observatory, Pino Torinese, Italy (proposed by L. J. Comrie) ;
Charles Mowat, Hall Cottage, St. Mary's Holm, by Kirkwall, Orkney (proposed by F. J. Hargreaves) ;

Francis John Sellers, M.I.M.E., 42 Church Crescent, London, N. 10 (proposed by F. J. Hargreaves);
Henry Frederic Tiarks, Foxbury, Chislehurst, Kent (proposed by T. E. R. Phillips);
George Mortimer Townsend, Hawera, New Zealand (proposed by P. O'Dea); and
Alfred Vowles, Bampton Street, Minehead, Somerset (proposed by W. F. Denning).

Seventy-one presents were announced as having been received since the last Meeting.

Corrections to the Mass of Venus. By J. K. Fotheringham, D.Litt.

Dr. Brown has discussed * my first paper on "Trepidation," so far as I deduce therein from observations of the Sun an apparent change of 1 or 2 per cent. in the ratio of the mass of Venus to that of the Sun coincident with the major changes in the apparent fluctuations of the longitude of the Moon. He does not refer to my discussion of the observations of Mars, which, though they were assigned a smaller total weight than the observations of the Sun, confirmed the general character of the fluctuation and formed the only evidence for the markedly small value of the mass of Venus in the twentieth century. Without this confirmation from the observations of Mars I do not think I should have ventured to affirm a real fluctuation in the mass of Venus.

1. *Dr. Brown's Gravitational Objection.*—Dr. Brown gives the form of an equation of condition for observations of the Sun,

$$\delta L + M' . \delta\mu' + E . \delta e + \Pi . e\delta\varpi = \delta\epsilon,$$

and states that Newcomb solves the equations of condition in groups of about eight years. His groups do, as a fact, vary in length, but the Greenwich observations from 1803 to 1882 are solved in groups of four years only. In consequence the second largest term passes through $2\frac{1}{2}$ revolutions only in each group, and there is the possibility of an error being introduced through the inclusion of an odd half revolution. There is also a possibility of error arising from the unequal distribution of the observations. As the distribution of the observations bears no relation to the periods of the Venus terms, these causes of error disappear if we combine a sufficiently large number of Newcomb's group results. The most dangerous source of error is, probably, that the form of equation used by Newcomb—and the same will apply to those used by Dr. Cowell, Dr. Ross, Hough, and Dr. Jones—takes no notice of the possibility of a progressive change in the observed as compared with the tabular longitude. The corrections to the mass of Venus ex-

* *M.N.*, **87** (1927), 524–7.

pressed by $\delta\mu'$, to the eccentricity expressed by δe, and to the longitude of perihelion expressed by $\delta\varpi$, depend on the differences between the observed longitude at opposite phases of their coefficients, M′, E, and Πe, respectively. These are all either periodic terms or sums of periodic terms. Any progressive change in the observed as compared with the tabular longitude will produce a systematic difference between the values of $\delta\epsilon$ at opposite phases, and will therefore affect the determination of $\delta\mu'$, δe, and $\delta\varpi$, unless the distribution of the observations over the different phases of these periodic terms and the order in which the different phases are grouped are, by accident or design, such as to prevent the attribution of progressive changes in mean longitude to these unknown quantities. If the observations are equally distributed throughout a group, and if the group contains an exact number of periods, then this source of error will be at its greatest where a periodic term, expressed as a sine term, completes its positive values before beginning its negative values, or *vice versa*, *i.e.* when its phase is 0° or 180° at the beginning of the group, and the source of error will be eliminated when positive and negative phases precede and follow one another with equal frequency, *i.e.* when the initial phase is either 90° or 270°. Since each group begins at the beginning of January, this source of error will have its full effect on the determination of the eccentricity; but, if the observations are evenly distributed, it will not affect the determination of the longitude of perihelion. A four-year group contains almost exactly five periods of the largest Venus term, two and a half of the second largest, and one of the third largest. The largest and third largest will, in consequence, go through their phases in the same order in any one group as in each of the other groups in the series so long as the system of four-year groups is maintained. The determined mass of Venus may, therefore, be expected to fluctuate backwards and forwards, according as the progressive change in the observed as compared with the tabular longitude is in a positive or a negative direction. Such a progressive change may be due to an error in the adopted value for the mean motion of the Sun at epoch, to an omission to allow for an acceleration of the Sun's motion, to a long-period fluctuation not regarded in the tables, or to the adoption of a false coefficient for a term of long or middle period sufficient to have an appreciable effect on the Earth's motion meaned over four years. A glance at the table printed by Newcomb on p. 22 of his *Astronomical Constants* will show a marked sixteen years' periodicity in the values, not only of the mass of Venus, but of the other terms determined throughout the eighty years in which the Greenwich observations are analysed in four-year groups. This is probably due to an error in the assumed mass of Mars, which produces a term with a period of fifteen years nine months in the motion of the Earth. A similar sixteen years' periodicity may be observed in the reduction from Leverrier's to Newcomb's tables of the Sun in Dr. Ross's paper, "The Sun's Mean Longitude."* Where, as in Newcomb's treatment of other observatories and of Greenwich outside the range of years specified, the observations are analysed in less systematically arranged groups,

* *Astronomical Journal*, **29**, 153, 154 (1916).

we may expect more erratic variations in the values determined by this method, and in any case the rhythm of the variations will be affected by irregularities in the distribution of the observations over the different groups.

In view of the source of error just discussed I can only give a qualified assent to Dr. Brown's assertion that "if $\delta\epsilon$ contain a term with very long period, it is easily seen from the numerical equations for any group that its effect is almost wholly confined to δL and that it exerts but little influence on the value of $\delta\mu'$." Apart from the years of bad observation in 1779–1838 and from the sixteen years' periodicity, the progressive change in the observed as compared with the tabular longitude is comparatively small in the Greenwich series discussed by Newcomb, though, as is well known, it assumes large dimensions in the present century. While, therefore, Dr. Brown's assertion cannot be applied to individual group results, it is approximately true when the groups are meaned so as to eliminate the sixteen years' periodicity, and, in particular, when contemporary observations grouped on different principles are compared with each other. Thus, when it is noted that every group which contains any part of the years 1867–73 gives a large mass to Venus, we may treat this as the observed result for that epoch. This will be found to be the case at Greenwich, Paris, Washington, Oxford, and Pulkova. Unfortunately the group determinations do not often display unanimity on this scale.

Dr. Brown's assertion that "to affect $\delta\mu'$ sensibly it is necessary to have changes in $\delta\epsilon$ with periods nearly equal to the periods of $g - g'$ or of $2g - 2g'$, that is to say, to periods of nineteen months and ten months respectively," is true in the sense that these are the only short-period changes that would produce this effect. His next sentence: "Hence, if a change in $\delta\mu'$ appears after we compare the solutions of the various groups, and if it be attributed to an actual change in μ' of a periodic character, this change must have a period nearly equal to that of $g - g'$ or to that of $2g - 2g'$," is inaccurately expressed. Such short-period changes in μ' would appear in $\delta\mu'$, but long-period changes in μ' would also produce changes in $\delta\epsilon$ with periods nearly equal to the periods of $g - g'$ and of $2g - 2g'$, though in this case they would also affect the determination of δL. Dr. Brown's argument, in which he shows that a fluctuation in μ' of nineteen months' period would produce a long-period term with a coefficient of 375″, has no relation to my discussion, in which I deduced a fluctuation in μ' of very long period. Dr. Brown's argument that the long-period fluctuation in μ' which I posited would produce a long-period term with a coefficient of 7″·2 in the longitude of the Sun, is substantially the same as what I stated in a letter published in *Nature* on February 26, in which I expressed myself as follows: "It is probably too early to give a final explanation of these interesting fluctuations, but it is clear that there is a fluctuation in the total action on the earth which determines its mean motion, amounting to between one-third and one-fourth of the fluctuation in the action of Venus but in the opposite direction, so that an increase in the action of Venus is accompanied by a diminution in the total action, and *vice versa*. This seems to imply something like

an interchange of mass between the Sun and Venus, and perhaps between the Sun and other planets." In deducing the ratio of one-third or one-fourth I had taken the coefficient of the major fluctuation in the Sun's longitude as 1″·4, and had obtained by differentiation the fluctuation in the total action on the Earth. The fluctuation in the action of Venus was taken from my papers on "Trepidation." The ratio between the constant terms produced by the two actions was assumed to be not greatly different from the ratio between the masses of the Sun and Venus respectively. I had not applied the factor 1·79 which appears in the addition to Dr. Brown's paper. While, therefore, I am in agreement with Dr. Brown as to the effect of a fluctuation in the mass of Venus unaccompanied by any change of gravitational action elsewhere in the solar system, it will be seen that I did not regard this as an objection to the acceptance of the observed fluctuation, but rather as an indication of the nature of the physical change of which that fluctuation is one phenomenon. The argument seems to show that any change in the mass of Venus must be mainly due to the acquisition or loss by that planet of matter which would in any case be somewhere within the Earth's orbit.

2. *Dr. Brown's Suggestion of Changes in Tabular Coefficients.*— Dr. Brown proceeds to offer a suggestion that the fluctuation in the determinations of the mass of Venus is due to changes in the coefficients of the Venus terms used in computing the tabular places. If the changes are regarded as unreal, they must be due to one of three causes: (1) Errors of observation; (2) errors in computing the ephemerides with which the observations are compared; (3) defects in the methods of analysis by which the terms are deduced. To the last of these causes I will recur later.

Dr. Brown chooses the second of these explanations. He supposes that some change has taken place in the coefficients of the Venus terms used in the tabular places, of which account has not been taken in the comparison of different determinations, and he draws attention to an apparent break about the year 1830 in the determinations given by Newcomb from observations of the Sun in his *Astronomical Constants*, pp. 22–24. This break is certainly significant, though there may be accidental causes which make it seem more significant than it is. While including the solar observations from 1779 to 1838 in my discussion, I excluded them from my numerical solution because of the extreme fluctuations which they gave for the motion of the Sun in longitude. Their value for the mass of Venus must be discounted, not only for this reason, but because of the discordant values which they give for that mass and because of their discordance from contemporary observations of Mars. 1830 is, however, approximately the epoch of a change in the Moon's motion. As will be seen from the table in Dr. Brown's paper in the *Yale Observatory Transactions*,* the Moon's motion in longitude from 1820·4 to 1830·6 was 0″·57 less than its mean value, while from 1830·6 to 1840·8 it was 3″·30 less than its mean value. If, therefore, the mass of Venus is connected with the rate of motion of the Moon, we should expect a change in the mass about this date.

* Vol. iii., part vi. (1926), 216.

Moreover, the marked increase in the mass of Venus which appears in Newcomb's table between the groups of 1827–30 and 1831–34 is one of a series of marked increases occurring at intervals of sixteen years. The group to group increases at these sixteen-year epochs as shown in Newcomb's table are as follows :—

Date.	Increase in Mass of Venus.*	Increase in Mean Longitude.	Increase in Ross's Reduction, Newcomb-Leverrier.
		″	″
1815	+·043	+1·34	−0·45
1831	+·061	+1·18	−0·23
1847	−·037	−0·25	−0·51
1863	+·027	+0·33	+0·02
1879	+·025	+0·07	−0·19

I am not able to offer an explanation of the apparent exception in 1847 which is found both in the mass of Venus and in the mean longitude. On the hypothesis which I advanced it would be reasonable to expect an increase in the mass of Venus about 1830, and it may be supposed that the sixteen years' periodicity has made this increase appear more abrupt than it really was. The break appears about 1845 in the Paris series and not at all in those of Cambridge and Königsberg. In the Greenwich series the mean correction to Leverrier's provisional mass of Venus is − ·03 from 1750 to 1830, and ·00, not + ·01 or + ·02 as Dr. Brown states, from 1831 to 1892. According to my formula, connecting the mass of Venus with the fluctuations of the Moon's longitude, the mean correction should have been − ·02 from 1750 to 1830, and ·00 from 1831 to 1892.

Dr. Brown quotes passages from Newcomb's *Astronomical Constants* to show that the tabular places used by him were not derived direct from published tables and that changes in his procedure occurred at 1830 and 1864. Of these the change at 1830 is alone of importance for our purpose. From 1750 to 1830 the mean correction found by Newcomb to the mass of Venus from the Greenwich series is − ·034, from 1831 to 1862 it is − ·002, and from 1863 to 1892 it is + ·003. Dr. Brown has, unfortunately, omitted two sentences from pp. 6, 7 of Newcomb's work which make clear what the change in his procedure at 1830 was. These sentences are : †

" As places of the Sun were necessary, not only for direct comparison with observations of the Sun, but also for the geocentric places of the planets, an ephemeris of the Sun's longitude and radius vector was prepared *for the entire period* 1750–1864 *to every fifth day,* the lunar perturbation being omitted and afterwards applied for each date when required. . . . After 1830 it was common to compute an ephemeris for intervals of three, five, or ten days, thus deriving the corrections necessary to reduce the published ephemerides of the *Berliner Jahrbuch* or of the *Nautical Almanac* to those derived from Leverrier's tables."

* Except where otherwise stated, Leverrier's provisional value of the mass of Venus is taken as unity throughout this paper.

† The italics in these quotations are mine.

With these we may compare a sentence on p. 20 referring to the Sun's Right Ascensions only:

"To lessen the number of equations of condition the residuals were divided into groups generally covering from ten to fifteen days, the length of the group being determined by the condition that the perturbations of Venus must not change much during the period."

These sentences show that both before and after 1830 Newcomb computed an ephemeris of the Sun's longitude for every fifth day. Before 1830 the Right Ascension for a particular day, if not computed directly, would be obtained by interpolation. From 1830 onwards the computed ephemeris was compared with the published ephemerides and the correction to the Right Ascension given in the published ephemeris would be obtained by interpolation. Both before and after that date the residuals thus obtained were meaned in ten- or fifteen-day groups. This was the only change in Newcomb's procedure at the date 1830. Now it is clear that a ten- or fifteen-day mean would not be appreciably affected, according as the daily places or the daily corrections to the published ephemerides were interpolated from the computed ephemeris. The only difference made by the change of procedure would be that more exact allowance would be made for the curvature in the perturbation by the latter procedure. But the argument of the largest Venus term goes through only 6° in five days, and the value of this term would be affected by less than $\frac{1}{2000}$ of itself by ignoring the curvature in 6°. The second term goes through 3° only and would be affected far less. As the observed change is $\frac{1}{31}$ of the value of the perturbations, the change of method fails as an explanation.

We have next to deal with the possibility of the difference between the coefficients actually used after Newcomb's tables were substituted for those of Leverrier in 1901 being far less than the difference between the masses on which they were supposed to depend. As one of Dr. Brown's quotations shows, Newcomb himself cast doubt on the accuracy with which he had reduced Leverrier's tabular places in order to accommodate them to the standard masses which Newcomb had decided to adopt. The only error in such a reduction which could give a systematically false coefficient to the Venus terms in the Sun's longitude would be an error in correcting the tables of the Sun for the mass of Venus. Had this correction always been applied accurately, the coefficients of these terms would have been reduced by $\frac{1}{251}$ of their tabular value. This is less than the difference between the separate determinations from the two principal Venus terms made by Dr. Cowell, and a numerical error within this limit is unimportant. And, if we turn to the observations of Mars, errors made by Leverrier cease to affect the question, for Dr. Ross,* as he explains, computed the geocentric place of Mars by Newcomb's tables for epochs separated by about two months, and thus obtained the necessary corrections to reduce the ephemeris of comparison to Newcomb's places. He states that these corrections were plotted and a smooth curve drawn through

* "New Elements of Mars," *Astronomical Papers of the American Ephemeris*, vol. ix., part ii. (1917), 257.

the points. The largest Venus term in the Earth's longitude goes through 75° in two months. If Dr. Ross had entirely ignored the curvature between one point and the next, he would have made an error of about one-thirteenth part of the correction to be applied to Leverrier's value of this term, or of about $\frac{1}{650}$ of the whole value of this term. In the case of the other terms the error would be far less. It is clear, therefore, that any systematic error in the reduction of the tabular places of Mars to the places of Newcomb's system cannot affect the determined mass of Venus by more than $\pm$ ·001, if by so much.

I have also compared Newcomb's and Leverrier's coefficients of the principal Venus terms in the Sun's longitude, and find that they are in the same ratio as the adopted masses. I have also computed from both sets of tables the total amount of the perturbations with periods equal to that of the difference between the longitudes of the Earth and Venus or of its multiples for certain selected dates, and find that the totals are approximately in the same ratio. In the case of Newcomb's tables I have confirmed this result by a trigonometrical computation of the inequalities for the same dates. There can, therefore, be no reason to suppose that any change in the coefficients has been inadvertently introduced by the substitution of Newcomb's tables for Leverrier's. The change is what it professes to be, neither more nor less. Incidentally I may remark that no error of the order required could have been produced if, as Dr. Brown suggests, Newcomb, in the MS. tables which he used for observations from 1750 to 1863, had contracted the coefficients of the Venus terms by taking them to one decimal place only. The coefficients of the two largest terms as they enter into Leverrier's tables are $-4''{\cdot}933$ and $+5''{\cdot}635$ respectively. Dr. Brown's hypothesis would mean that at 1864, when Newcomb began to use the tables without previously contracting the coefficients, he had unwittingly added + ·007 to the mass used in the ephemeris of comparison, and we should have to suppose that the apparent increase of + ·005 in the mass about that date corresponded to a real increase of + ·012, the opposite of what Dr. Brown desires to prove.

For these reasons I consider that Dr. Brown's attempt to explain the apparent fluctuation in the mass of Venus by concealed changes in the coefficients of the Venus terms fails so far as the work of Newcomb and Dr. Ross is concerned.

3. *Discussions by Leverrier, Dr. Cowell, and Dr. Jones.*—The determination of the mass of Venus by Leverrier in the introduction to his tables of the Sun * may be taken as the starting-point of modern discussions. Leverrier forms from a large mass of observations a series of equations connecting the corrections to the eccentricity of the Earth's orbit, the longitude of its perihelion, and the masses of Mercury, Venus, and Mars. Separate equations were formed for the two groups 1750–1810 and 1811–1850. Provisional values of the mean place and mean motion of the Earth had been previously obtained and were assumed in the formation of these equations. The determination of the mass of Venus is, unfortunately, vitiated by an error of sign. In

* *Annales de l'Observatoire de Paris,* tome iv. (1858), especially pp. 93–96.

reducing equation (v) on p. 94 of his discussion to form equation (vi) on p. 95, Leverrier inadvertently changed the sign of the term dependent on the mass of Mars which is given as $-\ 0''{\cdot}17\nu'''$ instead of $+\ 0''{\cdot}17\nu'''$. In consequence of this he obtained, as the correction to his provisional value of the mass of Venus from the observations of 1811–1850, $+\ 0{\cdot}009 - 0{\cdot}003\nu$ instead of $+\ 0{\cdot}013 - 0{\cdot}003\nu$, where ν is the correction to the mass of Mercury. From the observations of 1750–1810 he had found, as the correction to the mass of Venus, $-\ 0{\cdot}001 + 0{\cdot}005\nu$. Ignoring the correction to the mass of Mercury, he found as the correction from all observations of the Sun $+\ 0{\cdot}004$, and adopted this value in his tables of the Sun. If we correct the error of sign, we get $+\ 0{\cdot}006$. In his tables of Mars and the outer planets, Leverrier substituted the correction $-\ 0{\cdot}025$, which he derived from a later study. What is of most interest to us is that Leverrier's study leads to values as wide apart as $-\ 0{\cdot}001$ and $+\ 0{\cdot}013$ from the two groups of solar observations, and that his error of sign is a precedent, which may excuse similar errors in more recent investigators.* It may be of interest to remark that Leverrier's two groups yield a still greater difference for the mass of Mars, the change being in the same direction for both planets.

I find that Dr. Cowell † has by an oversight attributed both to Leverrier and to Newcomb coefficients for the Venus terms other than what those astronomers intended, and that his work is vitiated in consequence. Leverrier in his theory, *Annales de l'Observatoire de Paris*, tome iii. (1857), p. 84, and tome iv. (1858), p. 34, obtains terms in the Sun's longitude

$$-\ 4''{\cdot}913 \sin (l'' - l') + 5''{\cdot}613 \sin (2l'' - 2l'),$$

where l'' is explained as signifying the mean longitude of the Earth referred to the equinox of 1850 and l' the mean heliocentric longitude of Venus referred to the same equinox. Dr. Cowell cites the formula as

$$-\ 4''{\cdot}913 \sin (\mathrm{V} - \mathrm{E}) + 5''{\cdot}613 \sin 2(\mathrm{V} - \mathrm{E}).$$

The dates which Dr. Cowell gives for certain specified values of p show that he must have intended V to mean the mean heliocentric longitude of Venus and E to mean the mean longitude of the Earth, thus inverting the meaning of Leverrier's symbols. This is equivalent to an error of sign in both terms.

In his tables Leverrier multiplies these terms by 1·004, and Dr. Cowell gives the product, again with false sign.

Newcomb does not express these inequalities in terms of the difference in longitude, but in terms of the difference in mean anomaly. Dr. Cowell has, however, made the same error of sign in converting Newcomb's terms to a form with the same argument as Leverrier's. These errors of sign do not vitiate his determination of the corrections to be applied to Leverrier's terms. They only vitiate his values for the

* Leverrier's error of sign produces, through the mass of Venus, an error in the second decimal figure in his value for $2e\delta\varpi$, and in the second decimal of a second of arc in his value for the centennial motion of the Sun in longitude. But it is not likely that Leverrier would have attached any importance to these figures.

† *M.N.*, **66** (1906), 305–7.

terms to be corrected. But, since the correction to the mass is found from the ratio between the correction to the coefficient and the coefficient itself, an error of sign in the latter involves an error of sign in the correction to the mass.

We must therefore substitute the following values for the ratio of the mass of Venus to that of the Sun for those given by Dr. Cowell :—

From the term in $V - E$, 0·994 to 400246.
„ „ $2V - 2E$, 1·000 to 400246.
Mean, 0·997 to 400246 or 1 to 401250.

This is equivalent to a correction of + ·001 instead of + ·007 to Leverrier's provisional value.

I may incidentally remark that on the middle of p. 305 in Dr. Cowell's paper " 1874 " is clearly an error for " 1876."

Dr. Jones's " Discussion of the Greenwich Observations of the Sun, 1836–1923," * which includes a determination of the mass of Venus, is characterised by a number of errors which affect his argument in different degrees.

In applying the correction for equinox Dr. Jones uses Mr. Thackeray's reductions of clock-star lists from 1836 to 1848 to the system of the Greenwich Five-Year Catalogue for 1890,† and applies a further correction, which appears to be of the form $+ 0^s{\cdot}049 + 0^s{\cdot}098T$ (1890), to reduce to Newcomb's system N_1. The centennial term $0^s{\cdot}098T$ is that given by Newcomb as the centennial correction to reduce the A.G. system to his system N_1.‡ But the correction for 1846 is given as $+ 0^s{\cdot}079$ instead of $+ 0^s{\cdot}066$. The corrections for equinox from 1849 onwards are taken from a condensed note by Dr. Hedrick, published by Dr. Brown.§ Dr. Jones has not realised that Dr. Hedrick gave the constant term only, and that it is necessary to apply a centennial term in order to find the proper correction for equinox at each date. I have examined the comparisons of the different systems given in the introductions to the several Greenwich catalogues, so far as they deal with the catalogues of 1860 and later dates, and I find that, with the exception of the ten-year catalogue of 1890, they can be satisfactorily reduced to Newcomb's system if we apply the constant correction cited by Dr. Hedrick and Dr. Jones from Newcomb, together with a term of $0^s{\cdot}001t$, where t represents the interval in years from the epoch of each catalogue. I can speak with less certainty about the ten-year catalogue of 1890, but a correction of $+ 0^s{\cdot}054 + 0^s{\cdot}0003t$ (1900) seems to approximate to the proper correction. In this case only Dr. Hedrick's constant correction applies, not to the epoch of the catalogue, but to a date ten years later. For the years 1903–1905 the ten-year catalogue was used with a correction of $+ 0^s{\cdot}054$. If my formula is correct, the total correction at the end of 1905 would be $+ 0^s{\cdot}002$. From 1906 to 1914 Newcomb's equinox was used. From 1915 onwards a correction of $+ 0^s{\cdot}005$ has to be applied.

* *M.N.*, **86** (1926), 426–38. † *M.N.*, **56** (1896), 466–71.
‡ *Astronomical Papers of the American Ephemeris*, vol. viii. (1898), 12.
§ *M.N.*, **73** (1913), 699, 700.

I find, however, that differences, never exceeding $\pm 0^s.003$, arise from different methods of comparing systems. Ignoring the years 1847 and 1849–55, where there are discordances between Mr. Thackeray's corrections and those deduced by Dr. Hedrick, the error resulting from Dr. Jones's neglect of the centennial term reaches its maximum at the close of the year 1888, when his observed longitudes require a correction of $+ 0^s.017$ or of $+ 0''.25$.*

Dr. Jones compared the observed places with Newcomb's tables for the years 1836–63 and 1901–23, but used the Greenwich comparison with the *Nautical Almanac* places based on Leverrier's tables for 1864–1900. In reducing these to Newcomb's theory he decided to ignore the differences in the terms of long period. According to Newcomb † these terms in the sense Leverrier *minus* Newcomb have an approximate value of $- 0''.33 + 0''.30T$ (1850). The correction necessary to remove the combined result of the faulty correction for equinox and the neglect of differences in the long-period terms fluctuates from 1864 to 1900 in the sense Newcomb *minus* Observed between $+ 0''.23$ and $- 0''.05$, the maximum change at one step being $0''.15$.

There are also errors in the reduction from Leverrier's tables to Newcomb's for the elliptic motion of the Earth. Dr. Jones has overlooked the fact that while Leverrier tabulates among the other planetary terms those planetary perturbations which have for their argument the mean longitude of the Sun, Newcomb includes these in the elliptic motion.‡ Dr. Jones has, therefore, in effect included these both among the planetary perturbations and in the elliptic motion for the years 1864–1900. In consequence Dr. Jones's differences require the following corrections in the sense Leverrier *minus* Newcomb :—

Perihelion	$+ 1''.4$
Principal elliptic coefficient in longitude	$+ 0.11$.
Mean anomaly	$- 1.6 + 0''.30T$.

In the last expression I include the correction for long-period terms. These corrections are of the same order as the corrections which Dr. Jones deduces from his discussion. I note further that in that discussion there appears to be some confusion between the theoretical values for the changes in eccentricity and in longitude of perihelion used by Newcomb in his tables and the values which he deduced from the observations. In one sentence Dr. Jones seems to imply wrongly that Leverrier also obtained his value for the motion of the perihelion from observation.

These erroneous reductions from Leverrier's tables to Newcomb's enter into the monthly means on which Dr. Jones's discussion of the mass of Venus is based. Here Dr. Jones uses Dr. Cowell's method, and repeats his errors of sign. But, whereas the error only arises in Dr. Cowell's discussion after the corrections have been obtained, it

* The same erroneous corrections for equinox are used by Dr. Brown in his paper on the "Longitude of the Moon," *M.N.*, **73** (1913), 700 ff., and by Mr. Glauert in his papers on the "Rotation of the Earth."

† *Tables of the Sun* (1895), p. 9.

‡ *See Astronomical Constants*, p. 180.

arises in Dr. Jones's discussion at the outset. He considered it necessary to apply to Leverrier's places from 1864 to 1900 the correction

$$+ 0''\cdot095 \sin (V - E) - 0''\cdot110 \sin 2(V - E)$$

in order to reduce them to the basis of Newcomb's tables.

Correcting his errors of sign, we have to substitute for this expression

$$- 0''\cdot095 \sin (V - E) + 0''\cdot110 \sin 2(V - E).$$

And we must therefore apply to each of Dr. Jones's errors (Tab. – Obs.) a correction of

$$- 0''\cdot190 \sin (V - E) + 0''\cdot220 \sin 2(V - E).$$

In the method used by Dr. Cowell and Dr. Jones V – E is represented as $= 180^\circ + p \times 360^\circ/19$, and p is supposed to increase by unity each month, but at intervals of eight years ten months the same value of p is assigned to two consecutive months. I find that Dr. Jones's p has each of the values 4, 5, and 13 twenty-four times, each of the values 14 and 15 twenty-five times, and each of its other values twenty-three times within the range of years 1864–1900. In order, therefore, to apply the correction just found and to reduce it to Dr. Jones's units of $0^s\cdot01$, it is necessary to multiply the correction by $100/15$, where $f = 23$, 24, or 25, according to the frequency with which the particular value of p occurs.

Ignoring the other errors and confining my attention to the erroneous reduction of the Venus terms, I form the following corrected table to take the place of Table III. in Dr. Jones's paper :—

Analysis of Terms in V–E *and* 2V–2E, 1836–1923.

V–E.	p.	Sum of Errors (Tab.–Obs.) (Unit $0^s\cdot01$).	No.	Mean (Unit $0^s\cdot01$).
198°·9	1	– 97	55	–1·76
217·9	2	– 48	55	–0·87
236·8	3	– 99	57	–1·74
255·8	4	– 84	56	–1·50
274·7	5	– 44	56	–0·79
293·7	6	– 96	56	–1·72
312·6	7	– 86	56	–1·54
331·6	8	– 82	55	–1·49
350·5	9	– 78	55	–1·42
9·5	10	–101	55	–1·84
28·4	11	– 39	55	–0·71
47·4	12	– 75	56	–1·34
66·3	13	– 88	56	–1·57
85·3	14	– 56	56	–1·00
104·2	15	–118	56	–2·11
123·2	16	–134	56	–2·39
142·1	17	–147	55	–2·67
161·1	18	–117	55	–2·13
180·0	19	– 61	55	–1·11

Analysing the mean residuals with Dr. Jones in the form

$$a + b \sin (V - E) + c \sin 2(V - E)$$

and converting from time to arc, I derive the following values for the coefficients of the sine terms :—

$$b = - 0''{\cdot}031 \pm 0''{\cdot}015$$
$$c = + 0''{\cdot}058 \pm 0''{\cdot}015.$$

In other words, the correction to the Venus terms adopted by Newcomb, viz.

$$+ 4''{\cdot}838 \sin (V - E) - 5''{\cdot}526 \sin 2(V - E)$$

is

$$+ 0''{\cdot}031 \sin (V - E) - 0''{\cdot}058 \sin 2(V - E)$$
$$\pm 0''{\cdot}015 \qquad \pm 0''{\cdot}015.$$

From each term separately a correction to Newcomb's adopted mass of Venus is derived. The term in sin (V − E) requires an increase in the mass in the ratio

$$1 + \frac{0{\cdot}031}{4{\cdot}838} \text{ to } 1 \qquad \text{or} \qquad 1{\cdot}0064 \pm 0{\cdot}0031 \text{ to } 1.$$

Similarly the term in sin 2(V − E) requires an increase in the ratio

$$1 + \frac{0{\cdot}058}{5{\cdot}526} \text{ to } 1 \qquad \text{or} \qquad 1{\cdot}0105 \pm 0{\cdot}0027 \text{ to } 1.$$

From the weighted mean we find

$$\Delta M/M = + 0{\cdot}0086 \pm 0{\cdot}0020$$

as compared with Newcomb, or

$$\Delta M/M = - 0{\cdot}0066 \pm 0{\cdot}0020$$

as compared with Leverrier's provisional value.

The corresponding value for the reciprocal of the mass of Venus is

$$404,\ 500 \pm 800.$$

It will be observed that although almost every number in the computation is transformed, the conclusion is almost identical with that at which Dr. Jones arrived. The reason for this is that the correction to Newcomb stands in the same ratio to the difference between Newcomb and Leverrier in which the number of years reduced with false sign from Leverrier's to Newcomb's system stands to the whole number of years in which Newcomb's terms appeared with a false sign.

4. *Greenwich Observations of the Sun*, 1901–1916.—One of the most striking defects in the material on which my discussion of the supposed fluctuation in the mass of Venus was based was that, whereas the Mars observations throughout and the Sun observations from 1750 to 1900 had been discussed in comparatively short groups of years, the Sun observations since 1900 had not been discussed except as part of Dr. Jones's eighty-eight year group, and there was therefore no direct evidence of the value that the Sun observations since 1900 would yield. I therefore thought it desirable to make a

separate analysis of the Greenwich meridian observations of the Sun from 1901 to 1916, that is, for the first sixteen years during which Newcomb's tables were used as an ephemeris of comparison. I had three reasons for choosing a sixteen-year period for analysis: (1) In order to eliminate any possible sixteen-year periodicity; (2) because in sixteen years both the principal Venus terms go through an exact number of periods, 10 and 20 respectively; (3) because during the particular sixteen years there were no bends in the trepidation curve, and, if this was connected with the mass of Venus, there was the expectation of a uniform mass.

In this discussion I have followed Dr. Cowell's method with certain modifications. I have used his auxiliary angle p with the same values that he has used. I have adopted for each month the mean error of right ascension as observed with the Greenwich transit circle, and I have omitted 1906 April and 1915 January, February, and March, when the transit circle was not in use, and 1906 January, when only one observation was made. I have also adopted $0^s{\cdot}001$ instead of $0^s{\cdot}01$ as my unit of error, and have applied to the observed errors (Tab. – Obs.) the correction $-0^s{\cdot}054$ for the years 1901, 1902, and $-0^s{\cdot}005$ for the years 1915, 1916, to reduce to the equinox of Newcomb's system N_1.

In my first analysis each month was assigned to that value of p which came nearest to the value of V – E on the mean date of the month's observations as given in the Greenwich Observations. The result was that the number of months to which the different values of p were assigned varied more than in Dr. Cowell's and Dr. Jones's investigations. With them p advances symmetrically, irrespective of the distribution of observations over the month, and gaps in the series are supplied by interpolation.

I thus got the following table :—

Analysis of Terms in V–E and 2V–2E, 1901–1916.

V–E.	p.	Sum of Errors (Obs.–Tab.).	No.	Mean (Unit $0^s{\cdot}001$).
198°·9	1	+472	10	+47·2
217·9	2	+316	10	+31·6
236·8	3	+358	10	+35·8
255·8	4	+485	10	+48·5
274·7	5	+610	11	+55·5
293·7	6	+266	10	+26·6
312·6	7	+385	10	+38·5
331·6	8	+442	10	+44·2
350·5	9	+524	10	+52·4
9·5	10	+507	11	+46·1
28·4	11	+349	8	+43·6
47·4	12	+335	9	+37·2
66·3	13	+387	9	+43·0

Analysis of Terms in V−E and 2V−2E, 1901–1916—*continued.*

V−E.	*p.*	Sum of Errors (Obs.−Tab.).	No.	Mean (Unit 0s·001).
85°·3	14	+480	11	+43·6
104·2	15	+422	10	+42·2
123·2	16	+470	9	+52·2
142·1	17	+536	10	+53·6
161·1	18	+521	11	+47·4
180·0	19	+305	8	+38·1

Two solutions were made from this table. In the one each month received equal weight. In the other each value of p received equal weight. In each solution I analysed in the form

$$a + b \sin (\mathrm{V} - \mathrm{E}) + c \sin 2(\mathrm{V} - \mathrm{E}).$$

The first solution gave

$$a = +0''\cdot654 \pm 0''\cdot018, \qquad b = +0''\cdot032 \pm 0''\cdot025,$$
$$c = -0''\cdot042 \pm 0''\cdot025.$$

The resultant corrections to the mass of Venus are:
From the term in V−E

$$+ \cdot0067 \pm \cdot0052$$

and from the term in 2V − 2E

$$+ \cdot0076 \pm \cdot0046.$$

The weighted mean gives a correction of

$$+ \cdot0072 \pm \cdot0034$$

to Newcomb's value, and of

$$- \cdot0080 \pm \cdot0033$$

to Leverrier's provisional value.

The second solution gave

$$a = +0''\cdot653 \pm 0''\cdot018, \qquad b = +0''\cdot035 \pm 0''\cdot025,$$
$$c = -0''\cdot043 \pm 0''\cdot025.$$

The resultant corrections to the mass of Venus are:
From the term in V − E

$$+ 0\cdot0073 \pm \cdot0051.$$

From the term in 2V − 2E

$$+ 0\cdot0078 \pm \cdot0045.$$

The weighted mean gives a correction of

$$+ \cdot0076 \pm \cdot0034$$

to Newcomb's value, and of

$$- \cdot0076 \pm \cdot0033$$

to Leverrier's provisional value.

The second solution involved far less labour, and the results are practically identical.

An analysis was then made, using for each month the same value of p as that adopted by Dr. Jones, omitting the same months as before. This involved the transference of 20 out of 117 monthly means to different values of p, and gave the values

$$b = +0''{\cdot}038, \qquad c = -0''{\cdot}046,$$

with resultant corrections to the mass of Venus :

From the term in V − E	+ 0·0079.
From the term in 2V − 2E	+ 0·0084.
Mean	+ 0·0082

as applied to Newcomb's value, or

$$-0{\cdot}0070$$

as applied to Leverrier's provisional value.

The result differs from the two former results by a small part only of the probable error. From this it may be inferred that extreme accuracy in assigning each monthly mean to its proper phase is unnecessary labour.

These solutions followed the usual practice in ignoring the possibility of a progressive change in the difference between tabular and observed longitude within the range of years examined. It was noticed that the observed longitude of the Sun had advanced + 1″·61 more than the tabular between 1901·0 and 1917·0. This was taken as equivalent to + 0ˢ·00056 per month, and a correction advancing by that amount was accordingly applied to each tabular monthly R.A., the correction to 1901 January being taken as zero. For this purpose Dr. Jones's values of p were employed. Means were formed as before. The results for each phase are shown in the following table :—

Revised Analysis of Terms in V−E and 2V−2E, 1901–1916.

V–E.	Mean Error (Obs.—Tab.) (Unit 0ˢ·001).	V–E.	Mean Error (Obs.—Tab.) (Unit 0ˢ·001).
198°·9	− 3·5	9°·5	− 2·9
217·9	−16·9	28·4	−19·8
236·8	−19·0	47·4	− 9·4
255·8	+ 0·4	66·3	−19·7
274·7	− 5·1	85·3	− 8·8
293·7	−19·5	104·2	−12·0
312·6	−18·9	123·2	−11·2
331·6	−10·8	142·1	+ 4·5
350·5	− 5·4	161·1	− 2·6
..	..	180·0	−12·5

The values now become

$$a = -0''\cdot152 \pm 0''\cdot018, \qquad b = +0''\cdot011 \pm 0''\cdot0255,$$
$$c = -0''\cdot032 \pm 0''\cdot0255.$$

The resultant corrections to Newcomb's mass of Venus are:
From the term in V − E

$$+ \cdot0023 \pm \cdot0053.$$

From the term in 2V − 2E

$$+ \cdot0059 \pm \cdot0046.$$

The weighted mean gives a correction of

$$+ \cdot0043 \pm \cdot0035$$

to Newcomb's value, and of

$$- \cdot0107 \pm \cdot0034$$

to Leverrier's provisional value.

This is probably as good a value as can be obtained from the evidence available. The close agreement between the two Venus terms, which is greater than that obtained from Dr. Cowell's or Dr. Jones's series, is noteworthy.

The probable error, though satisfactorily small, is greater than that derived from Dr. Jones's series where eighty-eight years were included instead of sixteen. The result is, however, discordant from the results found by Hough from the Cape heliometer observations of Mars from 1899 to 1920.* These gave as the correction to Leverrier's provisional value, $-\cdot027 \pm 0\cdot006$, if each opposition received equal weight, and $-\cdot046 \pm 0\cdot007$, if the observations at each opposition were weighted according to precision. The first of these differs from that just found from observations of the Sun by more than twice as much as we should expect from their respective probable errors. Dr. Jones, continuing the series to 1924, found $-\cdot026 \pm 0\cdot005$ from the whole series, weighting according to precision, but stated that the Greenwich meridian observations of Mars near opposition from 1901 to 1920 indicated a larger negative correction.†

Here we have the independent determinations from the two Venus terms in the longitude of the Sun combined against two independent series of observations of the geocentric longitude of Mars. Such a result is not due to observational error, nor can it well be due to an error in computing the ephemeris of comparison. It is probably due to the method of analysis adopted. Dr. Jones appears to have analysed both series of Mars observations by the same method, and good observations taken at the same dates and analysed in the same way naturally give similar results. It must be borne in mind that, though determinations from observations of Mars may be less affected by observational errors than determinations from observations of the Sun, they are more liable to be affected by systematic errors of analysis. As has been seen, Hough and Dr. Jones neither assumed nor analysed for any

* *Cape Annals*, viii., part vi. (1923), v, vii.
† *M.N.*, **85** (1925), 856, 857.

progressive change in the difference between observed and tabular longitude. Now, the effect of this omission on the determination of any term is in proportion to the length of its period. While the largest Venus term in the Sun's longitude has a period of nine and a half months, the largest Venus term in the longitude of Mars has a period of thirty-three years; and, since the Mars observations are confined to the neighbourhood of opposition, the Venus terms in the longitude of the Earth require several oppositions of Mars to pass through their various phases. Thus the largest Venus term in the Earth's motion appears with a phase diminished by about 120° at each successive opposition of Mars, and therefore takes three synodic periods of Mars or six and a half years to return to its original phase. There is, therefore, reason to expect that in the years analysed, during which the progressive change in the difference between observed and tabular longitudes was particularly rapid, a large part of this progressive change might in the analysis of the observations of Mars be attributed erroneously to the Venus terms.

I have also analysed the Greenwich observations of the Sun, 1901–1916, for a correction to the third Venus term. The analysis, in which the same progressive advance in the tabular R.A. was supposed as in the last preceding analysis, gave as the correction to Newcomb's coefficient of this term $+ 0''{\cdot}083$. As Newcomb's value of the coefficient is $+ 2''{\cdot}497$, this would imply a correction of $+ {\cdot}033$ to Newcomb's value of the mass of Venus. Since this term goes through four periods only in the years analysed as compared with twenty periods of the largest term and ten of the second largest, the result is less free from interference from terms not included in the discussion. It would also be affected to a greater extent by any discontinuity in the observations. I have thought it unnecessary to compute the probable error of this result.

5. *Greenwich Observations of the Sun,* 1867–1900.—The mass of Venus determined over the sixteen years, 1901–1916, differs little more than the probable errors would lead us to expect from that determined over the eighty-eight years, 1836–1923. In order to test the reality of the supposed fluctuation, it seemed desirable to analyse the preceding thirty-two years in two sixteen-year groups for terms in V − E and 2V − 2E. For this purpose I have used the errors tabulated by Dr. Cowell, applying the following corrections to his values (Tabular − Observed) to reduce to the equinox of Newcomb's system N_1:—

	Correction to Tabular R.A.		Correction to Tabular R.A.
	s		s
1867–69	−·03	1888–95	−·05
70–74	−·04	96–98	−·06
75–77	−·05	99–1900	−·05
78–87	−·04		

I have also accepted Dr. Cowell's interpolated value of the error in 1891 September.

In order to reduce to a minimum the effect of a progressive change in the difference between observed and tabular R.A., I have begun the series of observations for both terms at phase 90° approximately. The series in V − E begin, therefore, in 1869 January and 1884 December with $p = 15$, while the series in 2V − 2E begin in 1868 October and 1884 September with $p = 12$. Each series closes after sixteen years approximately with $p = 14$ for the term in V − E and $p = 11$ for the term in 2V − 2E.

As the series for the two terms are not quite identical, I give two values for mean error in each series for $p = 12, 13, 14$. The value given first in these cases refers to V − E, the bracketed value to 2V − 2E.

Analysis of Terms in V−E *and* 2V−2E, 1868–1900.

V−E.	*p*.	Mean Error, 1868–1884 (Obs.−Tab.) (Unit 0s·01).	Mean Error, 1884–1900 (Obs.−Tab.) (Unit 0s·01).
198°·9	1	+1·3	−2·9
217·9	2	+0·2	−3·2
236·8	3	+0·6	−2·0
255·8	4	0·0	−2·3
274·7	5	−0·8	−3·1
293·7	6	+0·7	−2·8
312·6	7	−0·2	−3·4
331·6	8	+0·2	−2·9
350·5	9	+0·3	−3·5
9·5	10	+0·2	−3·7
28·4	11	−1·4	−3·7
47·4	12	+0·2 (+0·4)	−2·5 (−2·3)
66·3	13	+0·6 (+0·2)	−3·1 (−2·4)
85·3	14	0·0 (+1·5)	−5·3 (−5·2)
104·2	15	+2·3	−4·7
123·2	16	+2·3	−5·1
142·1	17	+2·2	−2·2
161·1	18	+0·3	−5·0
180·0	19	−1·2	−3·1

The resultant values are:

	1868–1884.		1884–1900.	
	From V–E.	From 2V–2E.	From V–E.	From 2V–2E.
b	+0″·075 ±0″·030	−0″·075 ±0″·030	−0″·114 ±0″·028	+0″·073 ±0″·032
Correction to mass (Leverrier's tables).	+0·0153±·0061	+0·0133±·0054	−0·0231±·0056	−0·0129±·0057
Mean correction to mass (Leverrier's tables).	+·0142±·0040		−·0181±·0040	
Mean correction to mass (Leverrier's provisional value).	+·018 ±·004		−·014 ±·004	

It will be seen that while both terms in the observations from 1884 to 1900 are in good agreement with those from 1901 to 1916, there is a marked discrepancy between both results from the observations of 1868 to 1884 and those from the two later groups.

In order to test the apparent variation better, I examined the effect of making the first group begin either twenty months earlier or nineteen months later, retaining for each the same number of revolutions as before. I thus obtained the following results. The separate corrections to the mass of Venus refer to the value used in Leverrier's tables.

	1867–1883.	1870–1886.
Mass of Venus from term in V−E	+·0123	+·0037
„ „ „ 2V−2E	+·0104	+·0008
Mean correction to Leverrier's provisional value	+·015	+·006

The difference in the result made by pushing the whole series nineteen months later is very remarkable, and seems to imply a large difference between the mass in 1867–68 and that in 1884–86.

I also tried the effect of beginning the 1884–1900 series at the worst possible phases instead of at the most favourable. I therefore made the V − E series begin five months earlier and the 2V − 2E series three months later than in the analysis given above. This made the former series begin with phase 0° and the latter with phase 180°. I thus obtained from V − E and 2V − 2E respectively − ·0199 and − ·0109 as the corrections to the mass given in Leverrier's tables, as against − ·0231 and − ·0129 when both terms were made to begin with phase 90°. It would appear, therefore, that for this group the choice of initial phase is unimportant.

Solutions for the two groups, 1868–1884, 1884–1900, have also been made, with a more rigorous application of the correction for equinox. For this purpose the Greenwich catalogue for 1860 has been taken as standard, and the differences, Tab. − Obs., have been reduced to it by the application of the following corrections :—

	s		s
1870–77	−·006	1893–98	+·005
78–92	+·007	99–1900	0

The corrections to the mass used in Leverrier's tables become

	1868–1884.	1884–1900.
From V−E	+0·0137	−0·0231
From 2V−2E	+0·0138	−0·0138

The good agreement between the two terms for each group become still better, but the improvement is small compared with the probable error.

6. *Comparison of Different Kinds of Evidence.* — There appear to be three fairly well-marked chapters in the observed mass of Venus :

(i) A small mass, given by the Greenwich observations of the Sun, 1750–1830, the Paris observations of the Sun, 1801–44, and the

observations of Mars, 1751–1846. The Greenwich observations, discussed by Newcomb, as has been seen, yield − ·034 as the correction to Leverrier's provisional value for this epoch. The Paris observations give − ·026, and the Mars observations discussed by Dr. Ross give − ·009. With these we may compare Backlund's determination from Encke's comet for the years 1819–58, which gave − ·025.

(ii) A large mass, given by the Greenwich observations of the Sun, discussed by Newcomb, 1831–88, and by the same observations analysed in this paper for 1868–84, and found in both terms. A similar value is found in the Paris observations discussed by Newcomb, 1845–89, and in Dr. Ross's discussion of the Mars observations, 1847–75. Here Newcomb's Greenwich observations yield a mean correction of + ·002, the Paris observations yield + ·013, the Greenwich observations discussed above yield + ·018, and the observations of Mars + ·016.

(iii) A mass of intermediate value, but nearer to that of the first than of the second chapter. In this I include my determinations from the Greenwich observations of the Sun, − ·014 (1884–1900), and − ·011 (1901–16), together with Dr. Ross's determination, − ·016 from observations of Mars, 1877–1912, and Hough's and Dr. Jones' determinations from observations of Mars, 1899–1924, from which they deduced corrections ranging from − ·026 to − ·046.

Systematic observational error appears to be ruled out. The probable errors are a good guide to the effect of accidental observational error, and it seems quite clear that observational error will not explain the apparent fluctuation. The attempt to explain it by errors in computing the ephemerides is equally unsuccessful. Defects in the methods of analysis are certainly present, and are probably at their maximum since 1897, when the observed motions of the Sun and Mars have been most at variance with the tabular motions. But the main fluctuations are too great, too enduring, and too consistently attested to be explained in this way. It will be noted in particular that Newcomb's Paris groups are too irregular in duration and initial phase to be affected by the same errors. I see no other course open than to suppose the fluctuation real, but I observe that its phases do not correspond with the fluctuations in the longitude of the Moon and planets.

Summary.

1. It is shown that by the method of analysis commonly employed an apparent fluctuation of the mass of Venus might be due to a progressive change in the observed as compared with the tabular R.A. of the Sun. It is also shown that a long-period fluctuation in the mass of Venus would be represented as such by this method. Dr. Brown's demonstration that a fluctuation of nine and a half or nineteen months' period would give rise to a large periodic term is beside the point.

2. I reject Dr. Brown's suggestion that the tabular places are really dependent on other coefficients than those assumed in the analysis.

3. Leverrier's, Dr. Cowell's, and Dr. Jones's determinations of the mass of Venus are found to be affected by numerical errors, which are corrected. Other errors in Dr. Jones's paper, including his correction for equinox, are indicated.

4. An analysis of Greenwich observations of the Sun gives $+ \cdot 018 \pm \cdot 004$, $- \cdot 014 \pm \cdot 004$, and $- \cdot 011 \pm \cdot 003$ as the corrections to Leverrier's provisional mass of Venus for 1868–84, 1884–1900, and 1901–16. Each of these results is well supported by separate determinations of the terms in $V - E$ and $2V - 2E$.

5. A comparison of different groups of observations seems to indicate that there are three chapters in the history of the mass of Venus, which give a small mass from 1750 to *c.* 1830, a large mass from *c.* 1831 to *c.* 1888, and an intermediate mass, nearer to the small than to the large mass, from *c.* 1889 onwards. These chapters seem to be too well brought out to be explained by errors of observation or analysis, and it is concluded that they are real. They do not coincide with the observed fluctuations in the motion of the Moon.

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1927 *May* 2.
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Note on the Mass of Venus deduced from Cape Heliometer Observations of Mars. By H. Spencer Jones, M.A., Sc.D., H.M. Astronomer.

The observations of Mars obtained with the Cape Heliometer at each opposition from 1899 to 1924 were discussed in *M.N.R.A.S.*, **85**, 853, 1925, and corrections to Newcomb's orbital elements and adopted mass of Venus were derived. In the reductions it was assumed that the approximate formula

$$\Delta l = \frac{r - \mathrm{R}}{r} \Delta\lambda$$

was valid for observations secured near opposition, $\Delta\lambda$, Δl being corrections to the geocentric and heliocentric longitudes of Mars and r, R denoting the radii vectores from the Sun to the planet and the Earth respectively.

This formula implicitly assumes that the corrections required by Newcomb's Tables of Mars are much greater than those required by his Tables of the Sun, and that the latter can be neglected in comparison. In view of the fluctuations to which the longitude of the Sun in common with other bodies is subject, this assumption is not justifiable, and it is necessary that the assumed formula should be replaced by the following

$$\Delta l = \frac{r - \mathrm{R}}{r} \Delta\lambda + \frac{\mathrm{R}}{r} \Delta\mathrm{L},$$

where ΔL is the correction to the tabular longitude of the Sun. This formula holds for observations near opposition and in the case of Mars becomes

$$\Delta l - 0{\cdot}66\Delta L = \Delta\lambda\frac{r - R}{r} = \Delta l_1$$

where Δl_1 denotes the quantities tabulated previously (*loc. cit.*, p. 854) as Δl.

Δl, ΔL are both subject to fluctuations arising from irregularities in the rate of rotation of the Earth, and the simple method of treatment previously adopted is no longer valid. In addition, a correction to the mass of Venus adopted by Newcomb will affect the Venus terms in both Δl and ΔL; in the previous discussion, the Venus terms in Δl only were taken into account. The mass of Venus previously derived therefore needs revision.

Suppose, first, that the tabular longitudes are exact except for the fluctuations. Then, whether these arise wholly from irregularities in the rate of rotation of the Earth or in part from secular effects due to tidal friction, the effects on the longitudes are proportional to the ratio of the mean motions of the two bodies. Therefore, as the ratio of the mean motions of the Earth and Mars is 1·88, we have

$$\Delta L = 1{\cdot}88\Delta l \quad \text{or} \quad \Delta l = 0{\cdot}53\Delta L.$$

In general, the tabular longitudes will require corrections of the form $a + bt + ct^2$; in the case of Mars, the tabular longitude will further require correction on account of the errors of Newcomb's adopted eccentricity and longitude of perigee; both longitudes will also require correction for the error in the assumed mass of Venus. We may therefore write

$$\Delta\lambda \cdot \frac{r - R}{r} = \Delta l_1 = \alpha + \beta t + \gamma t^2 - 0{\cdot}13 f_\odot + B\cos l + C\sin l + \tfrac{1}{10}D(V_M - 0{\cdot}66V_\odot)$$

or

$$\Delta l_1 = A + A't_m + B\cos l + C\sin l - 0{\cdot}13 f_\odot + \tfrac{1}{10}D(V_M - 0{\cdot}66V_\odot).$$

The term in t^2 can be neglected for observations covering so short a period as those under discussion, provided that t is measured from the mid-epoch. $f_\odot$ denotes the fluctuation in the longitude of the Sun, unaccounted for by gravitational theory. V_M, $V_\odot$ denote the Venus terms in the longitudes of Mars and the Sun respectively.

The value of $f_\odot$ adopted for the purpose of discussion is necessarily based upon an assumed mean longitude which may be affected by error at epoch and by mean-motion error. The effect of such errors will be covered, however, by the terms $A + A't$. The values of $f_\odot$ which were used were taken from the paper by Dr. J. K. Fotheringham (*M.N.*, **87**, 147).

The values adopted for the eccentricity of Mars and for its longitude

of perigee were those derived in the previous discussion. We then have the following data :—

Year.	$\Delta l_1 + 0{\cdot}13 f_\odot$.	Computed.	O−C.	$\sqrt{p}$.
1899	+0″·04	−0″·09	+0″·13	7
1901	− ·15	·00	− ·15	11
03	+ ·19	− ·02	+ ·21	6
05	− ·05	+ ·03	− ·08	13
07	+ ·13	+ ·11	+ ·02	13
09	+ ·08	+ ·03	+ ·05	17
11	+ ·11	+ ·15	− ·04	12
14	+ ·32	+ ·25	+ ·07	10
16	+ ·29	+ ·25	+ ·04	10
18	+ ·24	+ ·25	− ·01	8
20	+ ·29	+ ·31	− ·02	7
22	+ ·32	+ ·19	+ ·13	12
24	+ ·18	+ ·20	− ·02	34

The second column gives the values of $\Delta l_1 + 0{\cdot}13 f_\odot$. Solving for A, A′, B, C, and D by least squares, with weights as indicated in the last column, the following values are derived :—

$$\begin{aligned} A &= +0''{\cdot}113 \pm 0''{\cdot}019 \\ A' &= +0{\cdot}305 \pm 0{\cdot}054 \\ B &= -0{\cdot}041 \pm 0{\cdot}027 \\ C &= +0{\cdot}015 \pm 0{\cdot}031 \\ D &= -0{\cdot}081 \pm 0{\cdot}034 \end{aligned}$$

The computed values and the residuals resulting from these elements are given in the preceding table. The unit of time for A′ is the interval of time from one opposition to the tenth following opposition, or about 21·4 years. With this solution, Σpv^2 is reduced to 11·0 as compared with 14·2 in the previous solution. A reduction would be expected as there is an additional unknown, but the probable error corresponding to unit weight is also reduced from 0″·83 to 0″·79.

The quantities A, A′ are not of great interest, being composite quantities involving errors of tabular longitudes at epoch and of mean motions for both Sun and Mars, and also of the corresponding quantities adopted for the Sun in deriving $f_\odot$.

The values of B and C lead to the following corrections to the eccentricity and perigee of Mars :—

$$\begin{aligned} \Delta e &= -0''{\cdot}002 \pm 0''{\cdot}015 \\ \Delta(\pi - \epsilon) &= +0''{\cdot}22 \pm 0''{\cdot}14 \end{aligned}$$

These corrections are applicable to the values derived in the previous discussion and the resulting corrections to Newcomb's elements become

$$\left.\begin{aligned} \Delta e &= +0''{\cdot}82 \pm 0''{\cdot}015 \\ \Delta(\pi - \epsilon) &= +0''{\cdot}75 \pm 0''{\cdot}14 \end{aligned}\right\} \text{Epoch 1916.}$$

On account of the longitude fluctuation, the correction to longitude at epoch $\Delta\epsilon$ cannot be derived separately.

The correction to the assumed mass of Venus (1/408,000) is given by

$$\frac{\Delta M}{M} = \tfrac{1}{10}D = -0{\cdot}0081 \pm 0{\cdot}0034$$

corresponding to a value for the reciprocal of the mass of

$$1/M = 411{,}300 \pm 1{,}400.$$

The correction is in the same sense as that derived in the preceding discussion, but of about two-thirds the magnitude.

The Equilibrium of the Calcium Chromosphere. By P. A. Taylor, Ph.D.

(*Communicated by R. H. Fowler, F.R.S.*)

1. The object of this paper is to modify Professor Milne's theory of the equilibrium of the calcium chromosphere † to take account of the curvature of the sun's surface and the variation of gravitational intensity with distance from the centre of the sun. In other respects the assumptions made will be the same as in Paper II., † pp. 15–19. Certain interesting points arise in the discussion of the chromosphere under these conditions. As might be expected by analogy with the theory of the isothermal gas-sphere, when the gravitation of the chromosphere itself is neglected the density is found to tend to a small positive constant at infinity. It will be found that this approximation is fully justified in the cases with which this paper is concerned, even at the earth's distance from the sun. Fortunately, the limiting value of the density is not impossibly great. A parameter μ' occurs, which plays a very similar rôle to that of μ, the coefficient of partial support, in Professor Milne's papers. The solution when $\mu' = 0$ is of special type, corresponding to the "fully supported" solution. The relation between μ' and μ is discussed, and it is found that μ' is always slightly less than μ. Formulæ are found for the density and for the proportion of ions in the excited state, both for $\mu' = 0$, and for $\mu' \neq 0$. These formulæ, which will be exhibited graphically for certain values of μ', will be used in a subsequent paper on the light-intensity of the calcium chromosphere.

I. *Fundamental Equations.*

2. In accordance with the assumptions already stated or referred to, we suppose that we have only to deal with light of one definite small frequency range $\Delta\nu$ about the frequency ν. Let $I_\nu^*(r, \theta)\Delta\nu$ be the intensity per unit solid angle subtended, per unit area of cross-section

† *Monthly Notices*, **84**, 354; **85**, 111; **86**, 8; **86**, 578. The last three of these papers will be referred to as Papers I., II., III. respectively.

of beam, of light of this small frequency range $\Delta\nu$ at a point at distance r from the centre of the sun, and in a direction making an angle θ with the outward-drawn radius vector through the point. Then by considering a small thin cylinder parallel to the radiation considered, we may show that the "$\cos\theta\frac{dI_\nu}{dx}$" of II. (8)† becomes replaced by

$$\left(\cos\theta\frac{\partial I_\nu^*}{\partial r}-\frac{\sin\theta}{r}\frac{\partial I_\nu^*}{\partial\theta}\right),$$

and hence we have in place of II. (8)

$$\frac{\cos\theta}{\alpha}\left(\frac{\partial I_\nu^*}{\partial r}-\frac{\tan\theta}{r}\frac{\partial I_\nu^*}{\partial\theta}\right)=n_2\frac{q_1}{q_2}(\sigma+I_\nu^*)-n_1I_\nu^* \quad . \quad (1)$$

where $\sigma=2h\nu^3/c^2$, n_1 and n_2 are the numbers of atoms per unit volume in the normal (1σ) and excited (1π) states respectively, q_1 and q_2 are the statistical weights of these states, and α is the atomic absorption coefficient.

The equation of radiative equilibrium II. (7) remains the same, viz.—

$$n_1\int_0^\pi I_\nu^*\sin\theta d\theta=n_2\frac{q_1}{q_2}\int_0^\pi(I_\nu^*+\sigma)\sin\theta d\theta \quad . \quad (2)$$

The integral of the constancy of total net flux (*cf.* II. (9)) now takes the form

$$\int_0^\pi I_\nu^*\sin\theta\cos\theta d\theta=\frac{C}{2r^2} \quad . \quad . \quad . \quad (3)$$

C being an integration constant ($4\pi^2C\Delta\nu$ is the total outward net flux over any sphere concentric with the sun).

Now multiply equation (1) by $\sin\theta d\theta$ and integrate with respect to θ between the limits 0 and $\frac{1}{2}\pi$. We have

$$\int_0^{\pi/2}\sin\theta\cos\theta\left(\frac{\partial I_\nu^*}{\partial r}-\frac{\tan\theta}{r}\frac{\partial I_\nu^*}{\partial\theta}\right)d\theta=-\alpha n\int_0^{\pi/2}I_\nu^*\sin\theta d\theta+\sigma n_2\alpha\frac{q_1}{q_2},$$

where

$$n=n_1-n_2(q_1/q_2) \quad . \quad . \quad . \quad . \quad (4)$$

Now

$$\int_0^{\pi/2}\sin^2\theta\frac{\partial I_\nu^*}{\partial\theta}d\theta=I_\nu^*(r,\tfrac{1}{2}\pi)-2\int_0^{\pi/2}I_\nu^*\sin\theta\cos\theta d\theta.$$

Therefore

$$\frac{1}{r^2}\int_0^{\pi/2}\sin\theta\cos\theta\frac{\partial}{\partial r}(r^2I_\nu^*)d\theta-\frac{1}{r}I_\nu^*(r,\tfrac{1}{2}\pi)=-\alpha n\int_0^{\pi/2}I_\nu^*\sin\theta d\theta+\sigma n_2\alpha(q_1/q_2) \quad (5)$$

Similarly

$$\frac{1}{r^2}\int_{\pi/2}^{\pi}\sin\theta\cos\theta\frac{\partial}{\partial r}(r^2I_\nu^*)d\theta+\frac{1}{r}I_\nu^*(r,\tfrac{1}{2}\pi)=-\alpha n\int_{\pi/2}^{\pi}I_\nu^*\sin\theta d\theta+\sigma n_2\alpha(q_1/q_2) \quad (6$$

† This symbol means equation (8) of Paper II. This notation will be used throughout this paper in referring to equations in Professor Milne's papers.

3. Following Professor Milne, we now employ the Schuster-Schwarzschild approximation. We replace $I_\nu^*(r, \theta)$ by its mean values, I_ν and I_ν', in the ranges of θ $(0, \frac{1}{2}\pi)$ and $(\frac{1}{2}\pi, \pi)$ respectively. As an approximation it seems natural to put

$$I_\nu^*(r, \tfrac{1}{2}\pi) = \tfrac{1}{2}(I_\nu + I_\nu').$$

Then we have approximately

$$\frac{1}{r^2}\frac{d}{dr}(r^2 I_\nu) - \frac{1}{r}(I_\nu + I_\nu') = 2a[-nI_\nu + \sigma n_2(q_1/q_2)] \quad . \quad (7)$$

and

$$\frac{1}{r^2}\frac{d}{dr}(r^2 I_\nu') - \frac{1}{r}(I_\nu + I_\nu') = 2a[nI_\nu' - \sigma n_2(q_1/q_2)] \quad . \quad (8)$$

Equations (2) and (3) become approximately

$$n(I_\nu + I_\nu') = 2\sigma n_2(q_1/q_2) \quad . \quad . \quad . \quad (9)$$

and

$$I_\nu - I_\nu' = C/r^2 \quad . \quad . \quad . \quad . \quad (10)$$

respectively. Adding equations (7) and (8) and using equation (10), we find

$$\frac{d}{dr}(I_\nu + I_\nu') = -2na C/r^2.$$

Now introduce a new variable u, defined by the equation

$$du = -\frac{na\,dr}{r^2} \quad . \quad . \quad . \quad . \quad (11)$$

The zero of u is at present left undefined. Then

$$\frac{d}{du}(I_\nu + I_\nu') = 2C,$$

$$I_\nu + I_\nu' = 2C(u - u_0) \quad . \quad . \quad . \quad (12)$$

u_0 being an arbitrary constant. Therefore, by equation (9),

$$\frac{n_2}{n} = \frac{q_2}{q_1}\frac{C}{\sigma}(u - u_0) \quad . \quad . \quad . \quad . \quad (13)$$

4. We now require the modified form of the equation of mechanical equilibrium. We introduce the proper inverse square law of gravitation, but neglect the gravitation of the chromosphere itself; we shall show later that the gravitation of the chromosphere itself is quite negligible at distances from the sun as great as that of the earth. Remembering that C/r^2 takes the place of F_ν in the equation of net flux, we obtain

$$\frac{dp}{dr} = -(n_1 + n_2)\frac{mga^2}{r^2} + \frac{n\pi a C\Delta\nu}{cr^2} \quad . \quad . \quad (14)$$

where p is the pressure, m the mass of the atom, g the intensity of

gravitation at the sun's surface, a the radius of the sun, and c the velocity of light.

Since

$$n_1 + n_2 = n + n_2[1 + (q_1/q_2)]$$
$$= n\left[1 + \left(1 + \frac{q_2}{q_1}\right)\frac{C}{\sigma}(u - u_0)\right] \quad . \quad . \quad (15)$$

we may write equation (14) in terms of u—

$$\frac{dp}{du} = \frac{mga^2}{a}\left[1 - \frac{\pi a C \Delta\nu}{mga^2c} + \frac{C}{\sigma}\left(1 + \frac{q_2}{q_1}\right)(u - u_0)\right].$$

The integral of this is

$$p - p_0 = \frac{mga^2}{a}\left(1 - \frac{\pi a C \Delta\nu}{mga^2c}\right)(u - u_0) + \frac{mga^2C}{2\sigma a}\left(1 + \frac{q_2}{q_1}\right)(u - u_0)^2 \quad (16)$$

where p_0 is an integration constant. Since the zero of u was left undefined, the constant u_0, which occurs in equation (12), is quite arbitrary. We choose u_0 to satisfy the quadratic equation in u_0—

$$p_0 = \frac{mga^2}{a}\left[1 - \frac{\pi a C \Delta\nu}{mga^2c}\right]u_0 - \frac{mga^2C}{2\sigma a}\left(1 + \frac{q_2}{q_1}\right)u_0^2 \quad . \quad (17)$$

Equation (16) now takes the simpler form—

$$p = \frac{\mu' mga^2}{a}u + \frac{mga^2C}{2\sigma a}\left(1 + \frac{q_2}{q_1}\right)u^2 \quad . \quad . \quad (18)$$

where

$$\mu' = 1 - \frac{\pi a C \Delta\nu}{mga^2c} - \frac{Cu_0}{\sigma}\left(1 + \frac{q_2}{q_1}\right) \quad . \quad . \quad . \quad (19)$$

μ' is a convenient parameter which plays a very similar rôle in this work to that of μ in Professor Milne's papers. As we shall see later, it is not exactly the coefficient of partial support. Nevertheless the form of equation (18) shows that $\mu' = 0$ will give rise to a solution of special form, just as $\mu = 0$ does in Professor Milne's work. We now proceed to a consideration of this special case.

II. *Case* $\mu' = 0$.

5. Equation (18) becomes

$$p = \frac{mga^2C}{2\sigma a}\left(1 + \frac{q_2}{q_1}\right)u^2 \quad . \quad . \quad . \quad (20)$$

To find the density law, we have (see II. (27))

$$p = 2(n_1 + n_2)k\mathrm{T}_0,$$

where T_0 is the temperature of the chromosphere. Now the statement that (n_2/n) is small compared with unity is in accordance with the conditions of the problem, for, conducting the equations on the assumption

that (n_2/n) is small, we do obtain small values of (n_2/n) in the solution.† Therefore approximately

$$p = 2nk\mathrm{T}_0 \quad . \quad . \quad . \quad . \quad . \quad (21)$$

$$n = \frac{mga^2\mathrm{C}}{4\sigma\alpha k\mathrm{T}_0}\left(1 + \frac{q_2}{q_1}\right)u^2$$

$$= -\frac{r^2}{a}\frac{du}{dr},$$

by equation (11). This integrates and gives

$$u = \frac{\lambda' r_0}{a^4}\frac{r}{r - r_0} \quad . \quad . \quad . \quad . \quad (22)$$

where

$$\lambda' = \frac{4\sigma k\mathrm{T}_0 a^2}{mg\mathrm{C}[1 + (q_2/q_1)]} \quad . \quad . \quad . \quad (23)$$

and r_0 is an integration constant. Therefore

$$n = \frac{\lambda'}{a^2\alpha}\left(\frac{r_0}{a}\right)^2\frac{r^2}{(r - r_0)^2} \quad . \quad . \quad . \quad (24)$$

As we should expect by analogy with the theory of the isothermal gas sphere, n tends to a small positive constant at infinity.

Equation (17) for u_0 contains the integration constant p_0. As this has not been determined, we are free to determine u_0 in terms of the physical conditions of the problem. We use the condition that I_ν' is zero for very large r (*e.g.* at the earth's distance from the sun). Combining equations (10) and (12), we have

$$\mathrm{I}_\nu' = \mathrm{C}[u - u_0 - (1/2r^2)].$$

Therefore

$$u_0 = \text{limiting value of } [u - (1/2r^2)]$$
$$= \lambda' r_0/a^4.$$

Substituting in equation (13), we get

$$\frac{n_2}{n} = \frac{\mathrm{C}\lambda' r_0 q_2}{\sigma a^4 q_1}\frac{r_0}{r - r_0} \quad . \quad . \quad . \quad . \quad (25)$$

6. For light of a neighbouring frequency ν' which is not absorbed,

$$\mathrm{I}_{\nu'} = (\mathrm{I}_{\nu'})_0(a/r)^2,$$

where $(\mathrm{I}_{\nu'})_0$ is the photospheric value of $\mathrm{I}_{\nu'}$, *i.e.* $\sigma/[\exp(h\nu/k\mathrm{T}) - 1]$.‡ Also, if κ denotes the ratio of the residual intensity in the line to the intensity in the neighbouring continuous spectrum,

$$\kappa = (\mathrm{I}_\nu/\mathrm{I}_{\nu'})_{r\ \text{large}}$$
$$= \frac{\mathrm{C}/r^2}{\sigma(a/r)^2/[\exp(h\nu/k\mathrm{T}) - 1]},$$

† See fig. 2.

‡ Paper I., p. 122; the photosphere is supposed to radiate like a black body at temperature T.

since I_ν' is small compared with I_ν when r is large. Therefore

$$C = \kappa\sigma a^2/[\exp(h\nu/kT) - 1] \quad . \quad . \quad . \quad (26)$$

By equations (10), (12), and (22) we have

$$I_\nu = C\left[\frac{1}{2r^2} + \frac{\lambda' r_0}{a^4}\frac{r}{r - r_0} - \frac{\lambda' r_0}{a^4}\right].$$

As in Paper I.,† we assume that when $r = a$, I_ν has the photospheric value $\sigma/[\exp(h\nu/kT) - 1]$. Thus

$$\frac{1}{\kappa a^2} = \frac{1}{2a^2} + \frac{\lambda' r_0}{a^4}\left[\frac{a}{a - r_0} - 1\right]$$

$$\left(\frac{r_0}{a}\right)^2 + \frac{(2 - \kappa)a}{2\lambda'\kappa}\left(\frac{r_0}{a} - 1\right) = 0 \quad . \quad . \quad . \quad (27)$$

Now

$$\frac{(2 - \kappa)a}{2\lambda'\kappa} = \frac{amq(2 - \kappa)[1 + (q_2/q_1)]}{8kT_0[\exp(h\nu/kT) - 1]}$$

$$= 412,$$

with the values $k = 1{\cdot}372 \cdot 10^{-16}$, $T_0 = 5000°$, $m = 40{\cdot}07 \times 1{\cdot}651 \cdot 10^{-24}$ gm., $g = 2{\cdot}73 \cdot 10^4$ gm. sec.$^{-2}$, $a = 6{\cdot}966 \cdot 10^{10}$ cm., $\kappa = 0{\cdot}11$, $(q_2/q_1) = 3$, $(c/\nu) = 3{\cdot}950 \cdot 10^{-5}$ cm., $c = 3.10^{10}$ cm. sec.$^{-1}$, $h = 6{\cdot}55 \cdot 10^{-27}$ erg. sec., $T = 6000°$.

Thus equation (27) has a large negative root, $(r_0/a) \fallingdotseq -412$, and a root nearly unity, $1 - \delta$ (say), where

$$1 - 2\delta - 412\delta = 0,$$

$$1 - (r_0/a) = \delta = 1/414 = 0{\cdot}00242 \quad . \quad . \quad (28)$$

Now in order that n shall be a finite decreasing function of r for $r > a$, we must have $0 < (r_0/a) < 1$.

Therefore we must take the root given by equation (28).

Since (r_0/a) differs from unity by so little, we may put $(r_0/a) = 1$, except where the difference between (r_0/a) and a quantity of order unity is concerned. Equations (24) and (25) take the forms

$$n = \frac{\lambda'}{a^2\alpha}\frac{r^2}{(r - r_0)^2} \quad . \quad . \quad . \quad . \quad (29)$$

$$\frac{n_2}{n} = \frac{C\lambda' q_2}{\sigma a^3 q_1}\frac{r_0}{r - r_0} \quad . \quad . \quad . \quad . \quad (30)$$

By equation (19), since $\mu' = 0$, we have

$$\frac{\pi\alpha C\Delta\nu}{mga^2c} = 1 - \frac{C}{\sigma}\frac{\lambda' r_0}{a^4}\left(1 + \frac{q_2}{q_1}\right)$$

$$= 1 - \frac{4kT_0}{mga}$$

† p. 126.

approximately. Now $\frac{4kT_0}{mga} = 2{\cdot}18 \,.\, 10^{-5}$. Neglecting this in comparison with unity, we have

$$\alpha\Delta\nu = \frac{mgc}{\pi\kappa\sigma}[\exp(h\nu/kT) - 1] \quad . \quad . \quad . \quad (31)$$

by equation (26), and relation II. (6) gives

$$\tau = \frac{(q_2/q_1)\kappa h\nu}{4mgc[\exp(h\nu/kT) - 1]} \quad . \quad . \quad . \quad (32$$

This is the same value of τ (*i.e.* $1{\cdot}80 \,.\, 10^{-8}$ secs.) as that obtained by Professor Milne in Paper II.† Supposing that $\Delta\nu$ corresponds to a line width of $\Delta\lambda$ Ångstroms, we find

$$\left.\begin{aligned} \alpha &= 5{\cdot}38 \,.\, 10^{-14}/\Delta\lambda \\ \frac{\lambda'}{a^2\alpha} &= 5{\cdot}56\Delta\lambda \\ \frac{C\lambda' q_2}{\sigma a^3 q_1} &= 1{\cdot}635 \,.\, 10^{-5} \end{aligned}\right\} \quad . \quad . \quad . \quad (33)$$

7. We now investigate the mass of the chromosphere in this, the worst case with which we shall deal. If ρ is the density, we have approximately

$$\rho_0 = mn = \frac{m\lambda'}{a^2\alpha}\frac{r^2}{(r-r_0)^2}.$$

$$4\pi\int_a^r \rho r^2 dr = \frac{4\pi m\lambda'}{a^2\alpha}\int_a^r \frac{r^4 dr}{(r-r_0)^2}$$

$$= \frac{4\pi m\lambda'}{a^2\alpha}[\tfrac{1}{3}r^3 + r_0 r^2 + 3r_0{}^2 r + 4r_0{}^3 \log_e(r-r_0) - (r_0{}^4/[r-r_0])]_a^r$$

$$= \frac{4\pi m\lambda' a}{\alpha}\left\{\tfrac{1}{3}[(r/a)^3 - 1] + (r/a)^2 - 1 + 3[(r/a) - 1] + 1/[1 - (a/r)] + 4\log_e\frac{(r/a) - 1}{1 - (r_0/a)}\right\}$$

approximately, provided that r/a is not nearly unity. We find that this expression has the value $2{\cdot}64 \,.\, 10^{-13}\Delta\lambda \times$ (sun's mass) when $(r/a) = 1000$, that is, the gravitation of the chromosphere itself is quite negligible at several times the earth's distance from the sun.

III. *Case* $\mu' \neq 0$.

8. We now proceed to investigate the case when μ' is not zero. As before, equation (21), we have approximately

$$p = 2nkT_0$$

† p. 9. *Cf.* I. (25).

Therefore, by equation (18),

$$n = \mathrm{P}u(1 + \mathrm{Q}u) \quad . \quad . \quad . \quad . \quad (34)$$

where

$$\left.\begin{aligned} \mathrm{P} &= \frac{\mu' mga^2}{2ak\mathrm{T}_0} \\ \mathrm{Q} &= \frac{\mathrm{C}}{2\mu'\sigma}\left(1 + \frac{q_2}{q_1}\right) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (35)$$

By equation (11)

$$du = -\frac{\mathrm{P}\alpha u(1 + \mathrm{Q}u)dr}{r^2},$$

$$\frac{\mathrm{P}\alpha}{r} = \log\frac{u}{1 + \mathrm{Q}u} + \text{const.}$$

$$u = 1/[\mathrm{A}\exp(-\mathrm{P}\alpha/r) - \mathrm{Q}] \quad . \quad . \quad . \quad (36)$$

where A is an integration constant. Therefore

$$n = \mathrm{AP}\exp(-\mathrm{P}\alpha/r)/[\mathrm{A}\exp(-\mathrm{P}\alpha/r) - \mathrm{Q}]^2 \quad . \quad (37)$$

and

$$\frac{n_2}{n} = \frac{\mathrm{C}q_2}{\sigma q_1}\left[\frac{1}{\mathrm{A}\exp(-\mathrm{P}\alpha/r) - \mathrm{Q}} - \frac{1}{\mathrm{A} - \mathrm{Q}}\right] \quad . \quad . \quad (38)$$

since u_0 is the limiting value of $[u - (1/2r^2)]$ as before.

9. Now by equations (10), (12), and (36),

$$\mathrm{I}_\nu = \mathrm{C}\left[\frac{1}{2r^2} + \frac{1}{\mathrm{A}\exp(-\mathrm{P}\alpha/r) - \mathrm{Q}} - \frac{1}{\mathrm{A} - \mathrm{Q}}\right],$$

and when $r = a$, $\mathrm{I}_\nu = \sigma/[\exp(h\nu/k\mathrm{T}) - 1]$. Moreover, as when $\mu' = 0$, $\mathrm{C} = \kappa\sigma a^2/[\exp(h\nu/k\mathrm{T}) - 1]$. Therefore

$$\frac{1}{\kappa a^2} = \frac{1}{2a^2} + \frac{1}{\mathrm{A}\exp(-\mathrm{P}\alpha/a) - \mathrm{Q}} - \frac{1}{\mathrm{A} - \mathrm{Q}},$$

$$f(\mathrm{A}) \equiv \mathrm{A}^2 - \left\{\frac{2\kappa a^2}{2 - \kappa}[\exp(\mathrm{P}\alpha/a) - 1] + \mathrm{Q}[\exp(\mathrm{P}\alpha/a) + 1]\right\}\mathrm{A} + \mathrm{Q}^2\exp(\mathrm{P}\alpha/a) = 0 \quad . \quad (39)$$

Now

$$f(\pm\infty) = +\infty$$

$$f[\mathrm{Q}\exp(\mathrm{P}\alpha/a)] = -[2\kappa a^2/(2 - \kappa)]\mathrm{Q}\exp(\mathrm{P}\alpha/a)[\exp(\mathrm{P}\alpha/a) - 1] < 0 \text{ for all } \mu'$$

$$f(\mathrm{Q}) = -[2\kappa a^2/(2 - \kappa)]\mathrm{Q}[\exp(\mathrm{P}\alpha/a) - 1] < 0 \text{ for all } \mu'.$$

Thus equation (39) in A always has two real roots, one less than Q, and one greater than $\mathrm{Q}\exp(\mathrm{P}\alpha/a)$. To decide which is the correct value of A, we use the criterion that n is a decreasing function of r. By differentiating equation (37) we get

$$\frac{dn}{dr} = -\frac{\mathrm{AP}^2\alpha}{r^2}\frac{[\mathrm{A}\exp(-\mathrm{P}\alpha/r) + \mathrm{Q}]}{[\mathrm{A}\exp(-\mathrm{P}\alpha/r) - \mathrm{Q}]^3}\exp(-\mathrm{P}\alpha/r).$$

This is to be negative for $r>a$. Therefore $A>Q \exp(P\alpha/a)$, that is, A is the greater of the two roots of equation (39).

We find the following numerical values :—

$$P\alpha = 6{\cdot}39.10^{15}\mu', \quad (P\alpha/a) = 9{\cdot}175.10^{4}\mu', \quad Q = 2{\cdot}54.10^{18}/\mu',$$
$$[2\kappa a^2/(2-\kappa)] = 5{\cdot}65.10^{20}, \quad (Cq_2/\sigma q_1) = 3{\cdot}81.10^{18} \quad (40)$$

10. In this paragraph $\mu' \geqslant 10^{-4}$ throughout, and then $\exp(P\alpha/a)$ is large compared with unity, and we may write the equation for A

$$A^2 - \beta A + \gamma = 0 \quad . \quad . \quad . \quad . \quad (41)$$

where

$$\beta = \{[2\kappa a^2/(2-\kappa)] + Q\} \exp(P\alpha/a)$$
$$\gamma = Q^2 \exp(P\alpha/a).$$

Also (γ/β^2) is very small compared with unity, and the greater of the roots of equation (41) in A is β approximately.

Therefore

$$A = \{[2\kappa a^2/(2-\kappa)] + Q\} \exp(P\alpha/a) \quad . \quad . \quad (42)$$

With this value of A we find that

$$\frac{Cu_0}{\sigma}\left(1 + \frac{q_2}{q_1}\right) = \frac{C}{\sigma(A-Q)}\left(1 + \frac{q_2}{q_1}\right) < 2.10^{-8}.$$

Therefore, by equation (19), unless μ' is very nearly unity, the value of $\alpha\Delta\nu$ is $(1-\mu')$ times its value when $\mu' = 0$ (equation (31)). Therefore we have

$$\alpha = 5{\cdot}38 \cdot 10^{-14}(1-\mu')/\Delta\lambda, \quad P = 1{\cdot}19.10^{29}\mu'\Delta\lambda/(1-\mu'),$$
$$PQ = 3{\cdot}015.10^{47}\Delta\lambda/(1-\mu') \quad . \quad (43)$$

We now know all the constants in the formulæ for n and n_2/n.

IV. *General.*

11. We now proceed to connect the parameter μ' with μ, the true coefficient of partial support. We may write equations (14) and (15)

$$\frac{(n_1+n_2)mga^2}{r^2} = \frac{nmga^2}{r^2}\left[1 + \frac{C}{\sigma}\left(1 + \frac{q_2}{q_1}\right)(u-u_0)\right] = -\frac{dp}{dr} + \frac{n\pi a C\Delta\nu}{cr^2}.$$

Now the left-hand side of this equation represents the weight of the atoms in unit volume. Therefore, if at distance r from the centre of the sun a fraction $[1-\mu(r)]$ of the atoms are supported by radiation pressure, we have

$$\frac{\pi a C\Delta\nu}{mgca^2} = [1-\mu(r)]\left[1 + \frac{C}{\sigma}\left(1 + \frac{q_2}{q_1}\right)(u-u_0)\right].$$

Now μ is the limiting value of μ (r), *i.e.* the value of μ (r) when $u = u_0$. Therefore by equation (19)

$$\mu = \mu' + \frac{Cu_0}{\sigma}\left(1 + \frac{q_2}{q_1}\right) \quad . \quad . \quad . \quad (44)$$

When $\mu' = 0$, by equation (22), $u_0 = (\lambda' r_0/a^4) = (\lambda'/a^3)$ approximately. Therefore

$$\mu = \frac{C\lambda'}{\sigma a^3}\left(1 + \frac{q_2}{q_1}\right) = 2{\cdot}18 \,.\, 10^{-5} \quad . \quad . \quad . \quad (45)$$

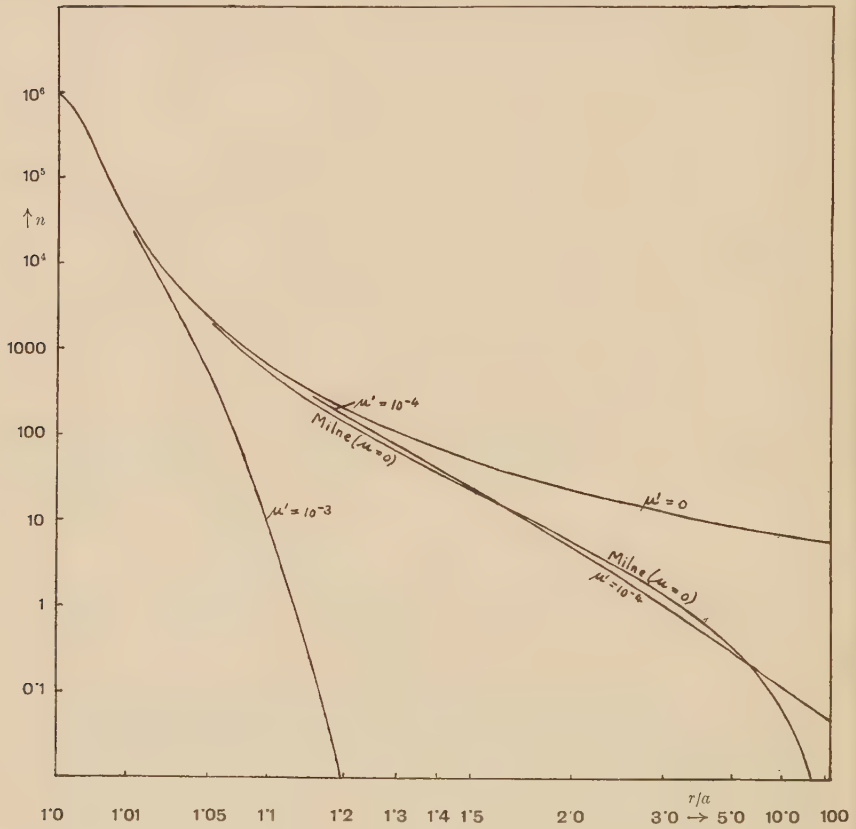


FIG. 1.—Density in atoms per cubic centimetre.

When $\mu' \neq 0$, by equation (36) $u_0 = 1/(A - Q)$, and

$$\mu = \mu' + \left\{\frac{C}{\sigma}\left(1 + \frac{q_2}{q_1}\right)\Big/(A - Q)\right\} \quad . \quad . \quad (46)$$

Since the last term in this equation is always positive, μ' is always less than μ.

When $\mu' \geqslant 10^{-4}$, the last term in equation (46) is less than $2 \,.\, 10^{-8}$, and therefore $\mu = \mu'$ to a close approximation.

Curves for n (or $[n_1 + n_2]$) and (n_2/n) (or $[n_2/(n_1 + n_2)]$) are drawn in

figs. 1 and 2 for $\mu' = 0$, 10^{-4}, 10^{-3} respectively. For purposes of comparison, curves have also been included drawn from Professor Milne's formulæ † when $\mu = 0$, formulæ obtained neglecting the curvature of the sun and the inverse square law of gravitation. In order conveniently to include a large range, log n and log (n_2/n) have been plotted against sec.$^{-1}$ (r/a), the scales being adjusted to read n, (n_2/n), and (r/a)

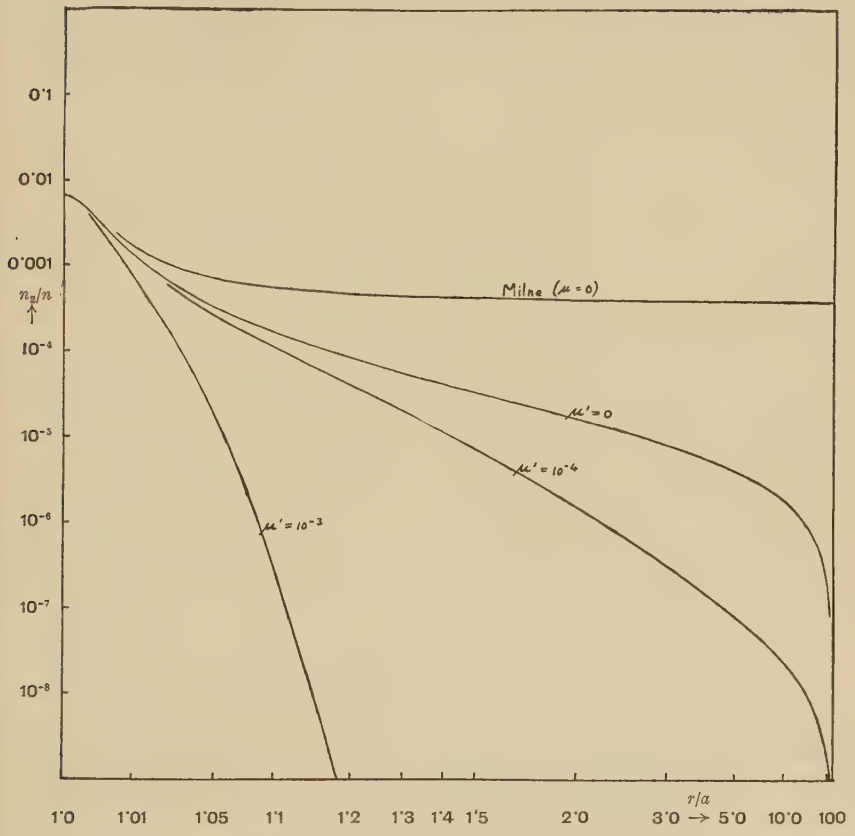


Fig. 2.—The proportion of atoms in the excited state.

directly.‡ We may note that in none of these cases is (n_2/n) greater than $6{\cdot}74 \cdot 10^{-3}$, thus justifying the assumption that (n_2/n) is small. Also we note that the limiting densities as given by our formulæ at great distances from the sun are $(5{\cdot}56,\ 0{\cdot}0474,\ 5{\cdot}5 \cdot 10^{-36}) \times \Delta\lambda$ atoms per

† The value of x_0 quoted in Paper I., p. 126, becomes halved approximately when account is taken of the statistical weights of the states.

‡ The curves for n have been drawn for the case $\Delta\lambda = 1$; for any other value of $\Delta\lambda$ the vertical scale must be considered raised or lowered through a suitable distance.

cubic centimetre according as $\mu' = 0, 10^{-4}, 10^{-3}$ respectively. These correspond to densities of $(3{\cdot}68 \cdot 10^{-22}, 3{\cdot}14 \cdot 10^{-24}, 3{\cdot}6 \cdot 10^{-58}) \times \Delta\lambda$ gm. cm.$^{-3}$ respectively.† We are tempted to try and link up with Professor Eddington's theory of interstellar matter. Unfortunately, however, our theory may break down at great distances from the sun, because there a large proportion of the calcium atoms may become doubly ionized,‡ and thus lose the support of radiation pressure. That is to say, beyond a certain distance from the sun the density may fall off more rapidly than is indicated by the formulæ of this paper.§

The Light-Intensity of the Calcium Chromosphere.

By P. A. Taylor, Ph.D.

(*Communicated by R. H. Fowler, F.R.S.*)

1. The results of the preceding paper are applied to the calculation of H and K radiation as observed in a telescope pointed near the limb of the sun in terms of the height above the limb of the point towards which the telescope is directed. The final integration can only be carried out numerically. This numerical integration is performed for $\mu' = 0, 10^{-4}$ (*i.e.* $\mu = 2{\cdot}18 \cdot 10^{-5}, 10^{-4}$). Comparison of the final result with observation indicates that the chromosphere is not exactly "fully supported"; the coefficient of partial support is probably of order of magnitude 10^{-4} (*i.e.* in the limit of great heights about 10^{-4} of the weight of the atom is balanced by the pressure gradient, the remainder of the weight being balanced by the radiation pressure).

† Such densities would have no effect on the earth's motion. In order to obtain cosmogonical effects, H. Jeffreys has to introduce a resisting medium of density $5 \cdot 10^{-14}$ or $4 \cdot 10^{-15}$ gm. cm.$^{-3}$ (*The Earth*, chap. iv.). The density for $\mu' = 0$ is rather high for interstellar matter (see A. S. Eddington, *Proc. Roy. Soc.*, A, **111** (1926), pp. 426, 428), but is not impossible, I think, for interplanetary matter.

‡ A. S. Eddington, *Proc. Roy. Soc.*, A, **111** (1926), pp. 438, 442.

§ Equations (9) (12), and the observed value of the residual intensity determine X, $= \int(n\alpha/r^2)dr$, or Y, $= \int n(a/r)^4 4\pi r^2 dr$, the integrals being taken from the photosphere to the earth's orbit. If n refers to Ca^+ atoms only, Y remains unaltered if Ca^{++} atoms are also present, for the mechanical relations are not used in determining Y. Also Y is always less than the total number of Ca^+ atoms, and if most of the chromosphere is near the sun, Y is approximately equal to the total number of Ca^+ atoms. That is to say, the observed residual intensity demands the presence of a certain amount of Ca^+, irrespective of how it arises. At great heights, where low densities prevail, there will be a continuous formation of Ca^{++} by expulsion of electrons by radiation from Ca^+ atoms, and if the rate of recapture of electrons by collision is less than the rate of expulsion of electrons, there will be a net continuous formation of Ca^{++} atoms; and presumably, in order to maintain a steady state, Ca^{++} atoms will fall back into the photosphere, and Ca^+ atoms will be expelled to take their place. The degree of ionization in this steady state will not necessarily be the same as in an equilibrium distribution of the same density and temperature. It is hoped to investigate these points in a subsequent paper.

I. *Calculation of Light-Intensity.*

2. We use the notation of the preceding paper. Equation (1) of that paper is

$$\frac{\cos\theta}{a}\left(\frac{\partial \mathrm{I}_\nu^*}{\partial r} - \frac{\tan\theta}{r}\frac{\partial \mathrm{I}_\nu^*}{\partial \theta}\right) = \sigma n_2(q_1/q_2) - n\mathrm{I}_\nu^* \quad . \quad (1)$$

Write $x = r\cos\theta$, $y = r\sin\theta$, so that y is the perpendicular distance from the centre of the sun to the beam of light considered, and x is the distance from the foot of that perpendicular along the beam in the direction of the light. Equation (1) becomes

$$\frac{\partial \mathrm{I}_\nu^*}{\partial x} + f(x, y)\mathrm{I}_\nu^* = g(x, y) \quad . \quad . \quad . \quad (2)$$

where

$$\left.\begin{aligned} f(x, y) &= an \\ g(x, y) &= a\sigma n_2(q_1/q_2) \end{aligned}\right\} \quad . \quad . \quad . \quad (3)$$

Integrating equation (2), we get

$$\mathrm{I}_\nu^* = \{\exp[-\int f(x, y)dx]\}\{\mathrm{F}(y) + \int g(x, y)\exp[\int f(x, y)dx]dx\} \quad (4)$$

where F is an arbitary function.

We are only concerned with beams such that y is greater than a, the radius of the photosphere. The condition for determining F is that I_ν^* tends to zero as x tends to negative infinity. So, with a change of integration variable, equation (4) takes the form

$$\mathrm{I}_\nu^* = \int_{-\infty}^{x} g(\xi, y)\exp\left[-\int_{\xi}^{x} f(\zeta, y)d\zeta\right]d\xi \quad . \quad . \quad (5)$$

We can obtain a formula for $\int f(\zeta, y)d\zeta$ when $\mu' = 0$, and an approximate formula when $\mu' = 10^{-4}$. The final integration is done numerically.

3. When $\mu' = 0$, we have by equations (29) and (30) of the preceding paper and equations (3) of this paper,

$$\left.\begin{aligned} f(x, y) &= \frac{\lambda'}{a^2}\frac{x^2 + y^2}{(\sqrt{x^2 + y^2} - r_0)^2} \\ g(x, y) &= \frac{\mathrm{C}\lambda'^2}{a^5}\cdot\frac{r_0(x^2 + y^2)}{(\sqrt{x^2 + y^2} - r_0)^3} \end{aligned}\right\} \quad . \quad . \quad (6)$$

If

$$\begin{aligned} \xi &= y\tan\phi \\ d\xi &= y\sec^2\phi\, d\phi, \end{aligned}$$

then

$$\mathrm{J}_y(\phi) = \int_0^{\xi} f(\zeta, y)d\zeta = \frac{\lambda' y}{a^2}\int_0^{\phi}\frac{\sec^2\phi\, d\phi}{[1 - (r_0/y)\cos\phi]^2}$$

and

$$\begin{aligned} g(\xi, y)d\xi &= \frac{\mathrm{C}\lambda'^2 r_0}{a^5}\frac{\sec\phi\, d\phi}{[1 - (r_0/y)\cos\phi]^3} \\ &= \frac{\mathrm{C}\lambda'^2}{a^4}\frac{\sec\phi\, d\phi}{[1 - (r_0/y)\cos\phi]^3} \end{aligned}$$

approximately. Then equation (5) becomes

$$I_\nu^* = \frac{C\lambda'^2}{a^4}\int_{\pi/2}^{\phi_0} \frac{\sec\phi}{[1-(r_0/y)\cos\phi]^3} \exp[J_\nu(\phi) - J_\nu(\phi_0)]d\phi \qquad (7)$$

where $\phi_0 = \tan^{-1}(x/y)$. If x is the earth's distance from the sun, $\phi_0 = 89^\circ\ 44'$. If x is sufficiently great for contributions to the integral from beyond an equal distance the other side of the sun to be negligible, we may write

$$\begin{aligned} I_\nu^* &= \frac{C\lambda'^2}{a^4}\int_{-\phi_0}^{\phi_0} \frac{\sec\phi}{[1-(r_0/y)\cos\phi]^3} \exp[J_\nu(\phi) - J_\nu(\phi_0)]d\phi \\ &= \frac{C\lambda'^2}{a^4}\int_0^{\phi_0} \frac{\sec\phi}{[1-(r_0/y)\cos\phi]^3} 2\{\exp[-J_\nu(\phi_0)]\}\cosh[J_\nu(\phi)]d\phi \qquad (8) \end{aligned}$$

On integrating out the expression for $J_\nu(\phi)$, we get

$$J_\nu(\phi) = H\log_{10}(\tan\phi + \sec\phi) + K\tan\phi + L\theta + M\sin 2\theta \qquad (9)$$

where

$$\tan\theta = N\tan\frac{\phi}{2} \qquad (10)$$

and

$$\left.\begin{aligned} H &= (2\lambda' r_0/a^2)\log_e 10 \\ K &= (\lambda' y/a^2) \\ L &= \frac{2(\lambda' y/a^2)(r_0/y)^2[3-2(r_0/y)^2]}{[1-(r_0/y)^2]^{3/2}} \\ M &= \frac{(\lambda' y/a^2)(r_0/y)^3}{[1-(r_0/y)^2]^{3/2}} \\ N &= \sqrt{\frac{1+(r_0/y)}{1-(r_0/y)}} \end{aligned}\right\} \qquad (11)$$

We find the following numerical values :—

(y/a).	1·00.	1·01.	1·05.	1·10.	1·20.
K	0·02085	0·02105	0·0219	0·0229	0·0250
L	125	11·3	1·55	0·683	0·326
M	62·0	5·31	0·617	0·228	0·0886
N	28·7	12·7	6·25	4·525	3·295

$$H = 0{\cdot}09575.$$

Also

$$(C\lambda'^2/a^4) = 7{\cdot}24\,.\,10^{-10}.$$

We can now perform the integration indicated in equation (8), $J_\nu(\phi)$ being given by equation (9). It is convenient to perform the integration in two stages. First we calculate I_ν^* on the assumption that the chromosphere is cut short at $r = 10a$. This means that in equation (8) we replace ϕ_0 by ϕ_0', where $\phi_0' = \sec^{-1}(10a/y)$. The calculations were made by Simpson's rule with 3° intervals of ϕ, except that in the cases $(y/a) = 1$ and 1·01, 1° intervals were used between $\phi = 0^\circ$ and $\phi = 30^\circ$. Secondly, the absorption factor due

to the part of the chromosphere between $r = 10a$ and the earth and the contribution of light from that part of the chromosphere will be very approximately the same for all the values of (y/a) considered. The absorption factor is found to be $\{1 - \exp[J_y(\phi_0') - J_y(\phi_0)]\}$, or $(1 - e^{-4\cdot4})$, when we put $(y/a) = 1$, $\phi'_0 = \sec^{-1} 10 = 84^\circ\ 16'$, $\phi_0 = 89^\circ\ 44'$. The contribution to I_ν^* from outside $r = 10a$ is

$$\frac{C\lambda'^2}{a^4}\int_{\phi=\phi_0'}^{\phi=\phi_0}\frac{[\exp\{J_y(\phi) - J_y(\phi_0)\}]_{(y/a)=1}\cos\phi\, d(\tan\phi)}{[1 - (r_0/y)\cos\phi]^3} \quad . \quad (12)$$

since the relevant values of $J_y(\phi)$ are very large, and this integral may be found graphically by plotting

$$\frac{\cos\phi[\exp\{J_y(\phi) - J_y(\phi_0)\}]_{(y/a)=1}}{[1 - (r_0/y)\cos\phi]^3}$$

against $\tan\phi$. (The integration is done this way because the "$\sec\phi$" factor makes the original integrand increase at a very rapid rate in the range considered.) The value of expression (12) is $4{\cdot}62 \cdot 10^{-10}$. Thus we find the values in the following table :—

(y/a).	$'I_\nu^*$.	I_ν^*.
1·00	$4{\cdot}02 \cdot 10^{-7}$	$5{\cdot}39 \cdot 10^{-9}$
1·01	$3{\cdot}84 \cdot 10^{-7}$	$5{\cdot}19 \cdot 10^{-9}$
1·05	$2{\cdot}73 \cdot 10^{-7}$	$3{\cdot}81 \cdot 10^{-9}$
1·10	$1{\cdot}49 \cdot 10^{-7}$	$2{\cdot}29 \cdot 10^{-9}$
1·20	$5{\cdot}95 \cdot 10^{-8}$	$1{\cdot}19 \cdot 10^{-9}$

$'I_\nu^*$ is the value obtained for I_ν^* when the chromosphere is assumed cut short at $r = 10a$.

$'I_\nu^*$ was introduced in order conveniently to divide the integration into two steps. But it was pointed out at the end of the preceding paper that beyond a certain distance from the sun, the density may fall off more rapidly than was indicated by the formulæ of that paper. It is possible, therefore, that $'I_\nu^*$ may better represent the intensity of radiation than I_ν^*.† It does not make much difference exactly where the chromosphere is cut short. For example, the contribution to $'I_\nu^*$ from outside the sphere $r = 5a$ is never as much as 2 per cent. of the whole for the values of (y/a) considered : the only appreciable difference that would result from supposing the chromosphere cut short at $r = 5a$, instead of $r = 10a$ would be that all the values of $'I_\nu^*$ would be increased by about 15 per cent., owing to the reduced absorption.

4. $\mu' = 10^{-4}$. We notice in fig. 1 of the preceding paper that the density laws for Professor Milne's treatment when $\mu = 0$ and the treatment of this paper when $\mu' = 10^{-4}$ are almost identical except when $r > 10a$ (say) ; and then the density is too small for the difference to matter. For example, the absorption between $r = 10a$ and $r = 215a$

† In the calculation of the true $'I_\nu^*$ for a cut short chromosphere, u_0 should strictly satisfy boundary conditions where the chromosphere is cut short, instead of in the limit of great distances. It may easily be shown that the effect of this is to increase $'I_\nu^*$ by a proportion varying from 0·8 to 2·0 per cent. as (y/a) increases from 1·00 to 1·20.

(the earth's distance from the sun) is approximately given by the assumption of an average density $\bar{n} = 0{\cdot}05\Delta\lambda$ for a distance of $200a$. The absorption factor is $[1 - e^{-0{\cdot}05 \times 200a \times a\Delta\lambda}]$, or $(1 - e^{-0{\cdot}04})$. The contribution to I_ν^* from atoms between $r = 10a$ and $r = 215a$ is approximately $a\sigma\bar{n}_2(q_1/q_2) \times 200a$, or of order 10^{-11}, if the mean value of (n_2/n) is taken as 10^{-7}. As I_ν^* turns out to be of order 10^{-7}, this contribution is negligible. So in calculating I_ν^* when $\mu' = 10^{-4}$, we may assume that n is given by Professor Milne's formula for $\mu = 0$, (n_2/n) being given, of course, by equation (38) of the preceding paper. Thus

$$f(x, y) = n\alpha = \frac{\lambda'}{(\sqrt{x^2 + y^2} - r_0)^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

$$g(x, y) = \sigma(q_1/q_2)(n_2/n)n\alpha$$
$$= C\left\{\frac{1}{A \exp(-P\alpha/\sqrt{x^2 + y^2}) - Q} - \frac{1}{A - Q}\right\}\frac{\lambda'}{(\sqrt{x^2 + y^2} - r_0)^2} \quad (14)$$

Putting, as before,

$$\xi = y \tan \phi,$$

we find

$$g(\xi, y)d\xi = C\lambda' y\left\{\frac{1}{A \exp[-(P\alpha/y)\cos\phi] - Q} - \frac{1}{A - Q}\right\}\frac{d\phi}{(y - r_0 \cos\phi)^2} \quad (15)$$

and

$$\int_0^\xi f(\zeta, y)d\zeta = \lambda' y \int_0^\phi \frac{d\phi}{(y - r_0 \cos\phi)^2}$$
$$= \frac{\lambda' y a}{(y^2 - r_0^2)^{3/2}}[(y/a)\theta + \sin\theta] \quad . \quad . \quad (16)$$

approximately, where

$$\tan(\theta/2) = \sqrt{(y + r_0)/(y - r_0)} \,.\, \tan(\phi/2) \quad . \quad (17)$$

As the formula we have used for n makes $n \to 0$ as $r \to \infty$, and as the integrals concerned are convergent, we may put $x = \infty$ in equation (5) and so

$$I_\nu^* = \frac{C\lambda' y}{r_0^2}\int_{-\pi/2}^{\pi/2}\left\{\frac{1}{A \exp[-(P\alpha/y)\cos\phi] - Q} - \frac{1}{A - Q}\right\}$$
$$\times \frac{\exp\{-[\lambda' y a/(y^2 - r_0^2)^{3/2}][(y/a)\theta + \sin\theta]_\theta^{\theta_0}}{[(y/r_0) - \cos\phi]^2} d\phi \quad (18)$$

where

$$\tan(\theta_0/2) = \sqrt{(y + r_0)/(y - r_0)} \,. \quad . \quad . \quad (19)$$

We may write equations (18) and (17)

$$I_\nu^* = 2B \exp\{-E[(y/a)\theta_0 + \sin\theta_0]\}$$
$$\times \int_0^{\pi/2}\left\{\frac{1}{A \exp(-F\cos\phi) - Q} - \frac{1}{A - Q}\right\}\frac{\cosh\{E[(y/a)\theta + \sin\theta]\}}{[(y/r_0) - \cos\phi]^2} d\phi \quad (20)$$

$$\tan(\theta/2) = S \tan(\phi/2) \quad . \quad . \quad . \quad (21)$$

where

$$B = (\lambda' C y/r_0^2), \qquad E = [\lambda' y a/(y^2 - r_0^2)^{3/2}],$$
$$F = (P a/y), \qquad S = \sqrt{(y + r_0)/(y - r_0)} \quad (22)$$

Now r_0 is the "r_0" of Professor Milne's second paper.† $[1 - (r_0/a)] = 0{\cdot}00263$. We find the values in the following table :—

(y/a).	1·00.	1·01.	1·05.	1·10.	1·20.
B	$1{\cdot}68 \cdot 10^{14}$	$1{\cdot}69 \cdot 10^{14}$	$1{\cdot}76 \cdot 10^{14}$	$1{\cdot}845 \cdot 10^{14}$	$2{\cdot}01 \cdot 10^{14}$
E	60·0	5·31	0·621	0·230	0·0843
F	9·175	9·08	8·74	8·34	7·645
S	27·6	12·6	6·32	4·02	3·29

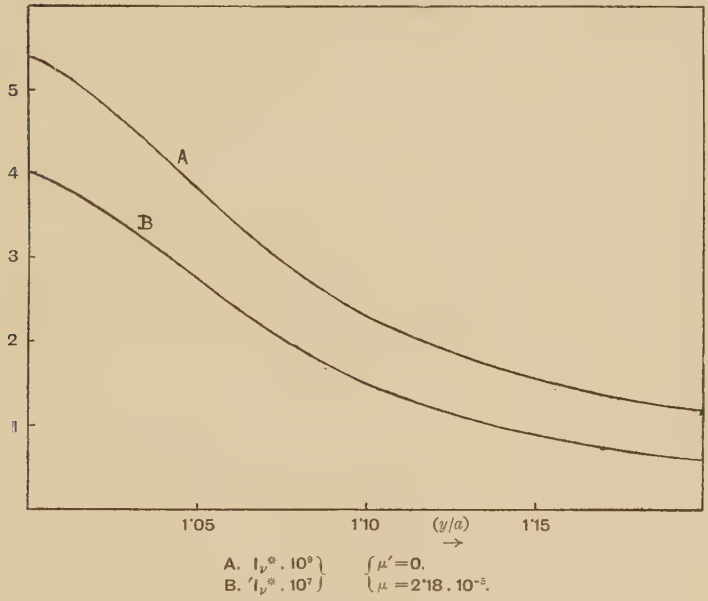


FIG. 1.

Also by equations (40) and (42) of the preceding paper,

$$Q = 2{\cdot}54 \cdot 10^{22}, \qquad A = 2{\cdot}505 \cdot 10^{26}, \qquad [1/(A - Q)] = 3{\cdot}99 \cdot 10^{-27}.$$

We can now perform numerically the integration indicated in equation (20). The calculations were made by Simpson's rule with 3° intervals of ϕ, except that between $\phi = 0°$ and $\phi = 30°$, 1° intervals were used for $(y/a) = 1{\cdot}00$ and $(y/a) = 1{\cdot}01$, $1\frac{1}{2}°$ intervals for $(y/a) = 1{\cdot}05$. In no case does any appreciable contribution to the integral

† See preceding paper, footnote †, p. 615.

come from the atoms outside $r = 2a$. We find the following values of I_ν^* :—

(y/a).	I_ν^*.
1·00	$4{\cdot}83 \cdot 10^{-7}$
1·01	$4{\cdot}83 \cdot 10^{-7}$
1·05	$3{\cdot}05 \cdot 10^{-7}$
1·10	$1{\cdot}17 \cdot 10^{-7}$
1·20	$2{\cdot}35 \cdot 10^{-8}$

We can now plot curves for I_ν^* and $'I_\nu^*$ when $\mu' = 0$, and for I_ν^* when $\mu' = 10^{-4}$.

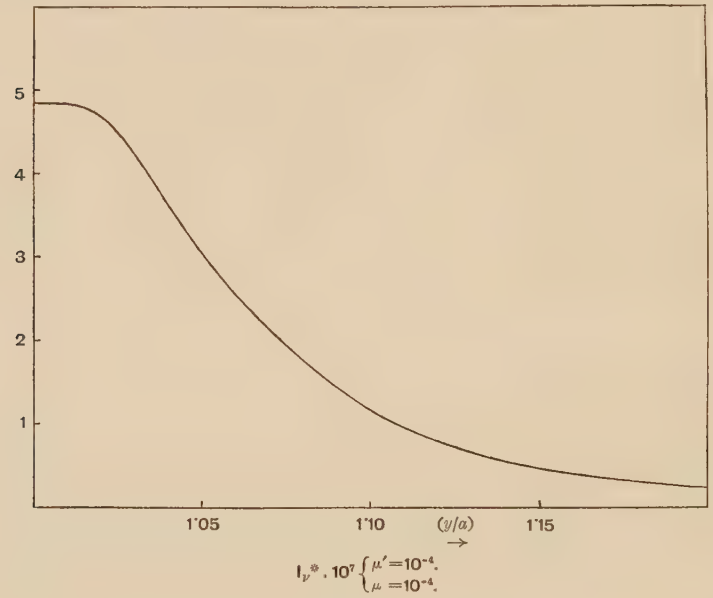


FIG. 2.

II. *Comparison with Observation.*

5. We now compare these curves with some recent eclipse observations.† Fig. 3 shows the moon's disc, of which M is the centre. S_1, S_2, and S are the positions of the centre of the sun's disc at second and third contact and at any intermediate time respectively. The interval between the two contacts was 190 seconds, and the plate was exposed from 20 seconds after second contact to 30 seconds before third contact. The slit was trained on TA from + 20 seconds to + 90 seconds, and on T′A′ from + 100 seconds to + 160 seconds (the zero of time being the instant of second contact). The total integrated intensity was measured

† F. J. M. Stratton, *Mem. R.A.S.*, **64**, 147.

at nine points P_r on TA (P_r' on T′A′). The experimental data are as follows :—

r.	TP_r.	$\int I_\nu * dt$	
		H Line.	K Line.
1	211″	39·6	61·3
2	234	33·9	49·0
3	277	31·0	600
4	320	49·6	116·2
5	363	1796	4000
6	423	27·7	42·4
7	445	22·2	38·4
8	488	11·2	20·6
9	552	..	5·8

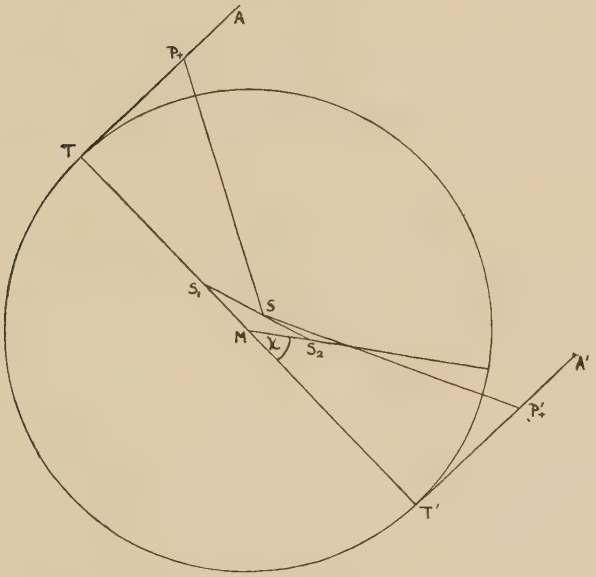


Fig. 3.

Readings 3, 4, 5 are certainly, and readings 6, 7 probably, affected by a solar prominence. The ninth K reading is obtained by extrapolation, but is probably not much greater than the value given, for the difficulty of measurement is due to the smallness of the reading. The values of $\int I_\nu * dt$ are only relative.

We proceed to calculate the values of $\int I_\nu * dt$ that would be given by the formulæ of this paper. We first calculate the ranges of SP_r and SP_r' corresponding to each reading. $\chi = 37{\cdot}2°$, $S_1M = 38''{\cdot}8$, $S_1T = 975''{\cdot}6$, the ratio $S_1S : S_1S_2$ ranges from $\frac{2}{19}$ to $\frac{9}{19}$ for first part of

exposure (P_r) and from $\frac{10}{19}$ to $\frac{16}{19}$ for second part of exposure (P_r'). By trigonometry we find the values in the following table :—

r.	$\frac{SP_r}{S_1T}$ ranges from	$\frac{SP_r'}{S_1T}$ ranges from
1	1·030 to 1·059	1·0615 to 1·038
2	1·035 „ 1·064	1·066 „ 1·0425
3	1·046 „ 1·074	1·076 „ 1·053
4	1 059 „ 1·086	1·088 „ 1·065
5	1·073 „ 1·0995	1·102 „ 1·0775
6	1·096 „ 1·1215	1·124 „ 1·100
7	1·104 „ 1·130	1·132 „ 1·108
8	1·123 „ 1·148	1·150 „ 1·127
9	1·154 „ 1·177	1·180 „ 1·156

$\frac{SP_r}{S_1T}$ and $\frac{SP_r'}{S_1T}$ will each change approximately uniformly during the exposure. We may therefore replace the time average of I_ν^* by its space average. This may be found by measuring the area under the appropriate segment of the appropriate curve in figures 1 and 2. If $\overline{(I_\nu^*)}_{(i)}$ and $\overline{(I_\nu^*)}_{(ii)}$ are the mean values of I_ν^* during the two parts of the exposure, we have

$$\int_{\text{whole exposure}} I_\nu^* dt = 70\overline{(I_\nu^*)}_{(i)} + 60\overline{(I_\nu^*)}_{(ii)} \quad . \quad . \quad . \quad (23)$$

t being measured in seconds. This is to be compared with the sum of the observed H and K integrated intensities. The mean values adopted are given in the following table :—

	r.	1.	2.	3.	4.	5.	6.	7.	8.	9.
$\mu' = 0$	$\overline{(I_\nu^*)}_{(i)} \cdot 10^9$	4·055	3·86	3·43	3·01	2·61	2·14	1·995	1·735	1·41
	$\overline{(I_\nu^*)}_{(ii)} \cdot 10^9$	3·835	3·64	3·285	2·89	2·525	2·08	1·945	1·685	1·39
$\mu = 2{\cdot}18 \cdot 10^{-5}$	$\overline{(I_\nu^*)}_{(i)} \cdot 10^7$	2·92	2·77	2·43	2·095	1·75	1·365	1·255	1·03	0·785
	$\overline{(I_\nu^*)}_{(ii)} \cdot 10^7$	2·75	2·59	2·305	1·99	1·695	1·315	1·215	1·005	0·775
$\mu' = \mu = 10^{-4}$	$\overline{(I_\nu^*)}_{(i)} \cdot 10^7$	3·40	3·11	2·50	1·91	1·51	1·025	0·915	0·705	0·445
	$\overline{(I_\nu^*)}_{(ii)} \cdot 10^7$	3·07	2·78	2·27	1·82	1·41	0·985	0·87	0·675	0·44

By means of equation (23) we find the following values of $\int I_\nu^* dt$:—

	r.	1.	2.	3.	4.	5.	6.	7.	8.	9.
$\mu' = 0$	$10^7 \int I_\nu^* dt$	5·14	4·89	4·37	3·84	3·34	2·75	2·56	2·225	1·82
$\mu = 2{\cdot}18 \cdot 10^{-5}$	$10^5 \int I_\nu^* dt$	3·69	3·49	3·08	2·66	2·24	1·74	1·61	1·32	1·01
$\mu' = \mu = 10^{-4}$,	$10^5 \int I_\nu^* dt$	4·22	3·845	3·11	2·43	1·90	1·31	1·16	0·898	0·575

As the observed values of $\int I_\nu^* dt$ are only relative, we compare the ratios

$$_rI_\nu = \frac{(\int I_\nu^* dt)_{r\text{th reading}}}{(\int I_\nu^* dt)_{1\text{st reading}}}.$$

We have the following values of $_rI_\nu$:—

r.		1.	2.	3.	4.	5.	6.	7.	8.	9.
Calculated $\mu' = 0$	$_rI_\nu$	1	0·95	0·85	0·75	0·65	0·53	0·50	0·46	0·35
Calculated $\mu = 2{\cdot}18 . 10^{-5}$	$_rI_\nu$	1	0·95	0·835	0·72	0·61	0·47	0·435	0·36	0·27
Calculated $\mu' = \mu = 10^{-4}$,	$_rI_\nu$	1	0·90	0·74	0·575	0·45	0·31	0·275	0·21	0·14
Observed . . .	$_rI_\nu$	1	0·82	[6·26	1·65	57·5	0·695	0·60]	0·315	(0·10)†

The observations which do not show signs of being affected by a solar prominence, *i.e.* 1, 2, 8, 9, on the whole favour $\mu' = 10^{-4}$ rather than $\mu' = 0$. It has already been pointed out that the ninth observed value, although doubtful, is not likely to be much greater than the value quoted. When $\mu' = 10^{-4}$ the errors are of both signs, whereas when $\mu' = 0$ the errors are all of one sign. At any rate μ is not likely to be much less than $2 . 10^{-5}$ (a lower value of μ means a less steep I_ν* curve). Also μ cannot be as great as 10^{-3}, for we see from figures 1 and 2 of the preceding paper that when $\mu' = \mu = 10^{-3}$, there is practically no chromosphere at all. Thus in the light of the rather scanty observational material available 10^{-4} seems a fair estimate of the order of magnitude of μ.

In conclusion I should like to thank Mr. Fowler for his kind advice and assistance at all stages of the work, and Mr. Stratton for supplying observational data in advance of publication.

Note on the Origin of Certain Radiations in Cometary Spectra.

By R. C. Johnson, M.A., D.Sc., King's College, London.

(*Communicated by Prof. A. Fowler, F.R.S.*)

In a paper by Lockyer (1) in 1912 on the spectrum of Comet Brooks, there is given a list of the principal radiations observed on photographs taken at South Kensington. The wave-lengths recorded by Lockyer are : two ultra-violet bands at about λλ3100 and 3370, together with a fainter band at λ3160. There are also strong condensations at λλ3883 and 4606–4737, a fairly strong group at about λ4050, and other fainter bands at about λλ4216, 4360, 5165, and 5635. Of these, the radiations at λλ5635, 5165, 4737, and 4360 were definitely identified with the Swan spectrum (which arises from a HC–CH molecule (2)), while groups at λλ3883, 4216, and 4606 are part of the well-known CN bands. With regard to the band at λ3370, the possibility of this being identified with the CN group about λ3360 is considered by Lockyer improbable, in view of the non-appearance of the CN group at λ3590. The origin of the three ultra-violet bands at λλ3100, 3160, and 3370, and of the strong condensation at about λ4050 was therefore unknown.

It is suggested that the bands at λλ3370 and 3160 are identical with the two bands at λλ3370 and 3159 in the second positive nitrogen band-

† From K line only.

spectrum, which is definitely known to arise from a neutral N_2 molecule. These two bands are the (0, 0) and (1, 0) bands * respectively, and are probably the two strongest bands in the whole system. We therefore have evidence of the presence of *free* nitrogen in the nuclei of comets. The group at λ4050 remains the outstanding problem of cometary spectra.

TABLE I.

Intensity.	λ.	Origin.
20	3885	CN
1	3970	
4	3994	
5	4020	
6	4049	?
4	4075	
3	4106	
1	4132	
3	4217	CN
5	4312	CH 4315
3	4370	Swan
1	4520	?
10	4690	Swan 4737 CN 4606
	etc.	

A photograph of the spectrum of Comet Brooks taken by Wolf at Heidelberg shows very similar features to the above. Measurements I have made from a contact positive (which is not a slit-spectrum) gave as the chief radiations those recorded in Table I. The probable origin of the various radiations is also given there. In addition to the unknown condensation at about λ4050, the presence of a fairly strong band at λ4312 is interesting. This may be identified with the CH band at λ4315 with considerable confidence. (This band is one of several which arise from a simple CH molecule, and which under certain conditions (2) are found associated with the Swan spectrum.)

Campbell has recorded measurements of an excellent slit-spectrogram of Comet 1907*d* (Daniel) in the *Lick Observatory Bulletin* (3), and the wave-lengths exhibit good agreement with those of other typical comets. Through the kindness of Professor Fowler I have been able to examine contact positives of Campbell's slit-spectrum of Comet Daniel, also of Comet 1893*b*, Gale's Comet 1894*b*, Swift's Comet 1899*a* and Halley's Comet 1910. These all exhibit radiations arising from the Swan bands, the λ4315 CH band, the violet CN bands, and the unknown group at about λ4050. In Campbell's spectrum, which shows much detail, there are bands at λλ3890 and 3874·1 which can probably be

* This is the usual notation in modern work on band-spectra. The band (n', n''), for example, results from a change in the vibrational quantum number $n' \rightarrow n''$.

identified with the heads of CH bands at about these wave-lengths. The bands at λλ4677·6 and 4670·6, though not identified by Campbell, are obviously the heads of the Swan bands (5, 4) and (6, 5). The only unidentified data, apart from about a score of comparatively faint lines which are recorded only in this particular spectrum, are those given in Table II., and constitute the unknown λ4050 group. The details are abstracted from Campbell's paper and may be compared with Table I.

Table II.

Comet 1907 *d*.	Comet 1893 *b*.	Comet 1894 *b*.	Comet 1899 *a*.
3990	3988	3987	3987
4003			
4014	4011	4015·7	4014
4020	4019·2	..	4019
4039	..	4040·9	
4042·8	4043·1	..	4042
* 4050·9	..	4050·5	
* 4052·9	4052·3	.	4052
4065·3			
* 4068·3	4069	4070·7	
* 4074	4075	..	4074

The four wave-lengths marked thus (*) were identified by Campbell with strong fine structure lines of the $\Delta n = n'' - n' = +1$ sequence. This identification is, however, very unlikely, for they are a considerable distance from the last observed head (about 90 A.U.), and their intensity is far too great to allow of their being explained in this way. The estimated intensities of Table I. make this clear: the λ4050 component of the group is several times as strong as the λ4216 CN head (the strongest head of the sequence). The persistence of the whole group, as well as the intensity distribution, suggests that it must be explained *as a whole*.

In a short paper (4) on Comet Mellish 1915 *a*, the suggestion is made that the group may be accounted for by a faint CH band discovered by Raffety (5) and in about the required position. This is, however, very unlikely in view of intensity considerations. The CH band at λ4050 is, taken as a whole, extremely faint compared to those at λλ4315 and $\left\{\begin{matrix}3890\\3874\end{matrix}\right.$, and it is therefore *very* improbable that the fine structure components of such a band could account for the strong cometary group.

I have examined another possibility, viz. that the so-called "tail" bands of CN may be the origin of the cometary group. This system of bands which can be produced either by the action of active nitrogen on carbon compounds, or by the carbon arc in nitrogen, has been recorded by Fowler and Strutt (6) and also by King (7). More recently (8) Mr. Jenkins and I have made new measurements of these bands as produced in active nitrogen. From this paper it will be seen that while there is a certain degree of overlapping of the "Tail" bands and the cometary

bands, the intensity distribution is very different in the two cases. The latter group is almost symmetrical in this respect about λ4050, while the maximum of intensity in the tail group is a little above λ3900. If the identity could be made it would involve a large energy displacement in the CN " tail " bands towards the less refrangible side. The possibility of such an energy displacement in these bands would seem evident on Mulliken's view of their origin (see a paper by Jevons (9)). If they are a development of the ordinary violet CN bands for a selection of very large values of n' and n'', then it is plausible that under suitable conditions of excitation the selected values of n' and n'' may be increased by 3 or 4 units. There are, however, numerous difficulties, into which it is unnecessary to enter here, in the way of accepting this view of the " tail " bands. I have attempted to effect an energy displacement in the " tail " system by developing them in the nitrogen afterglow in the presence of high pressure Helium (10), but without success. If this energy displacement can be realised experimentally, the precise agreement of the bands of the cometary spectrum with the " tail " bands will depend on reproducing the temperature of the cometary gases in the laboratory spectrum. Most of the " tail " bands are headless under low temperature conditions, and the location of the maxima of the P and R branches is, of course, determined by the temperature. It is these maxima which we should expect to agree with the cometary radiations. Until experimental evidence is available as to the possibility of this energy shift in the " tail " bands, we must still regard the origin of the λ4050 group an unsolved problem.

References.

(1) *Roy. Soc. Proc.*, A, **86**, 258, 1912.
(2) *Phil. Trans.*, A, **226**, 157–230, 1927.
(3) *Lick Observatory Bulletin*, **5**, No. 135.
(4) *Astrophysical Journ.*, **49**, 198, 1919.
(5) *Phil. Mag.*, **32**, 556, 1916.
(6) *Roy. Soc. Proc.*, A, **86**, 112, 1912.
(7) *Astrophysical Journ.*, **14**, 323, 1901.
(8) *Phil. Mag.*, VII, **2**, 625, 1926.
(9) *Roy. Soc. Proc.*, A, **112**, 425, 1926.
(10) *Roy. Soc. Proc.*, A, **102**, 453, 1923; *Phil. Mag.*, **48**, 1069, 1924.

On a Graphical Determination of the Orbital Elements of a Spectroscopic Binary. By J. K. E. Halm, Ph.D.

(Communicated by H.M. Astronomer.)

The graphical method here described of determining the elements of a spectroscopic binary is based on the following properties of the hodograph.

Suppose that a sufficient number of observations of radial velocities are available for constructing the velocity curve over one complete

period. In the usual way let this curve be constructed by plotting the radial velocities as ordinates against the dates of observation as abscissæ. If V_{max} and V_{min} represent the radial velocities at the maximum and minimum points of the curve, let a circle be drawn with radius $\frac{1}{2}(V_{max} - V_{min})$ whose centre lies $\frac{1}{2}(V_{max} + V_{min})$ above or below the axis of abscissæ. This circle represents the hodograph.

In fig. 1 let O represent the centre of this circle. Let a diameter be drawn making an angle ω (longitude of periastron) with the vertical. On this diameter let the distance EO be represented by ek, e being the eccentricity and k the radius of the hodograph. Further, let P_1Q_1 be

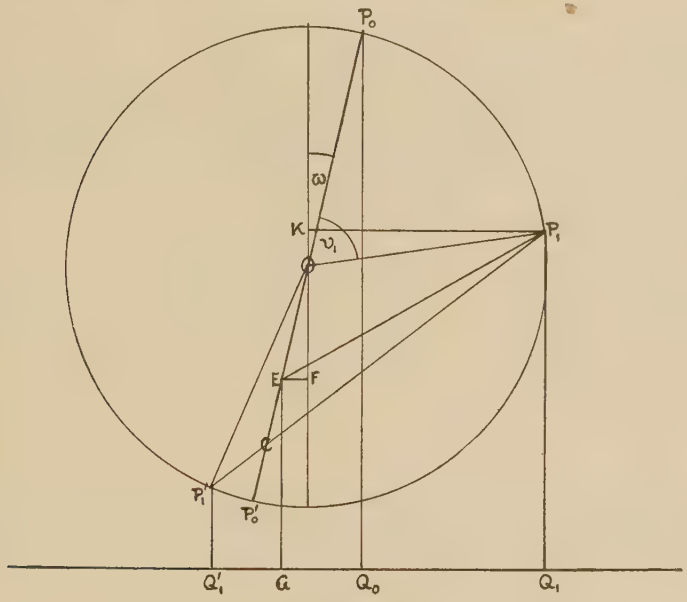


Fig. 1.

the radial velocity observed at the epoch t_1. The angle P_0OP_1 represents the true anomaly v. Hence

$$\begin{aligned} OK &= k\cos(v+\omega), \\ OF &= ek\cos\omega, \\ EG &= \gamma, \end{aligned}$$

γ being the radial component of the motion of the centre of gravity.

Hence

$$P_1Q_1 = k[\cos(v+\omega) + e\cos\omega] + \gamma,$$

which is the formula for the radial velocity at any given epoch. Further, if P_0Q_0 is the radial velocity at time t_0 (instant of passage through periastron), $(t_1 - t_0) \cdot \frac{360^\circ}{P}$ represents the mean anomaly at the instant t_1,

P being the period. The orbital velocity is represented in magnitude and direction by EP_1.

Obviously the elements of the orbit are known if ω and EO can be determined. For this purpose let another epoch t'_1 be considered which differs from t_1 by half a period. The radial velocity is supposed to be represented by $P'_1Q'_1$. The chord joining P_1 and P'_1 intersects the diameter $P_0P'_0$ in C. From a consideration of the triangles P_1OC and P'_1OC it is easily found that

$$OC = -k \cdot \frac{\cos \frac{1}{2}(v_2 - v_1)}{\cos \frac{1}{2}(v_2 + v_1)}.$$

Since v_1 and v_2 can be found from the given mean anomalies, OC can be calculated.

If this calculation is made for different chords $P_1P'_1$, $P_2P'_2$ . . ., the condition being maintained that the mean anomalies of the two points differ by half a period, it is found that for small eccentricities the values of OC/k are so nearly constant that the slight differences are negligible in a graphical construction. For instance, for $e = 0{\cdot}10$ the following values are obtained :

Mean Anomaly of			Mean Anomaly of		
P.	P'.	OC/k.	P.	P'.	OC/k.
(In fractions of period.)			(In fractions of period.)		
0·05	0·55	0·199	0·30	0·80	0·198
·10	·60	·198	·35	·85	·198
·15	·65	·198	·40	·90	·198
·20	·70	·198	·45	·95	·199
·25	·75	·198			

All the chords PP′ intersect therefore very nearly in the same point C on the diameter $P_0P'_0$. The direction of this diameter is thus fixed by the two points O and C, and hence ω is determined.

Further, it is found that in the case of small eccentricities, the point E lies midway between O and C. Hence

$$e = \tfrac{1}{2}OC/k.$$

Finally, γ is given by the length of the ordinate EG.

To show the simplicity and accuracy of the method, a fictitious orbit with the following elements was selected :—

$$\begin{aligned} t_0 &= 0{\cdot}00 \\ \omega &= 90^\circ \\ e &= 0{\cdot}100 \\ k &= 50\,\text{km/sec.} \\ \gamma &= 0 \\ P &= 100\ \text{days.} \end{aligned}$$

The time of passage through periastron was assumed at the beginning of the period. Any other epoch might, of course, have been chosen. Also,

for the sake of simplicity, the radial motion of the system has been assumed zero. The velocity curve calculated from the above elements is shown in fig. 2. The hodograph circle was then drawn, and on its circumference the radial velocities at the epochs 10, 20, etc. days were marked. Chords were next drawn through the points 0–50, 10–60, 20–70, etc. These intersect with great accuracy at the point C. Joining

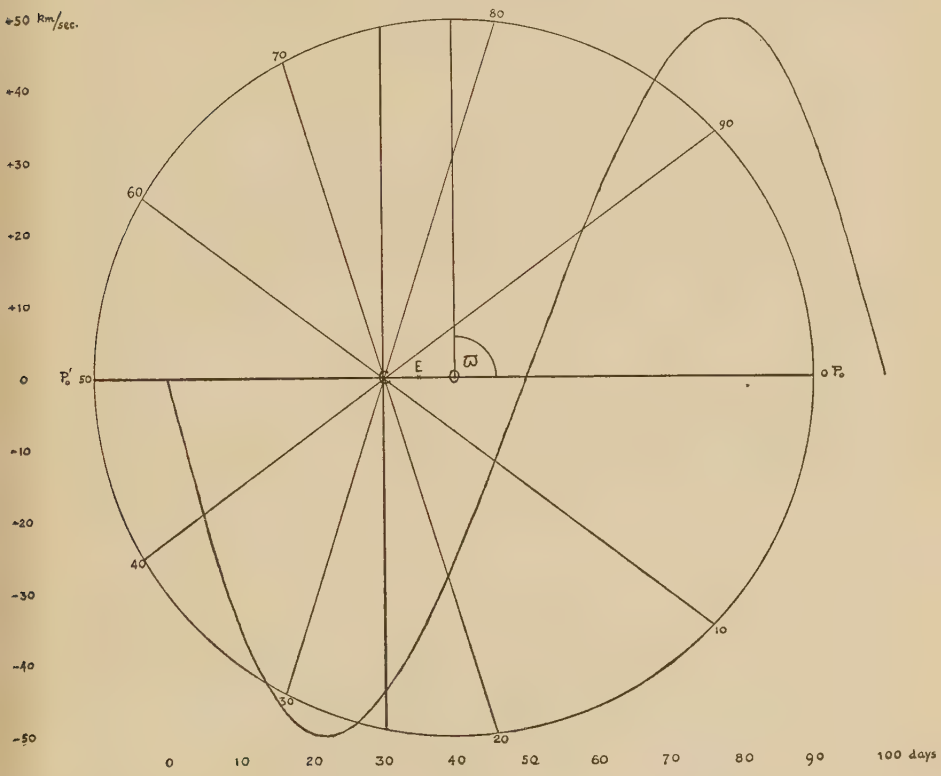


Fig. 2.

C and O, the horizontal diameter $P_0P'_0$ is obtained. Hence $\omega = 90°$, in agreement with the assumed value. Further,

$$CO/k = 10/50 = 0{\cdot}200,$$

and hence

$$e = \tfrac{1}{2}(CO/k) = 0{\cdot}100,$$

in agreement with the assumed value. Finally, the ordinate of E being zero shows that $\gamma = 0$.

For large eccentricities the condition

$$OC/k = \text{constant}$$

is less accurately satisfied. For instance, for $e = 0{\cdot}5$ the values are :

Mean Anomaly.			Mean Anomaly.		
P.	P′.	OC/k.	P.	P′.	OC/k.
0·05	0·55	0·791	0·30	0·80	0·771
·10	·60	·783	·35	·85	·776
·15	·65	·776	·40	·90	·783
·20	·70	·771	·45	·95	·791
·25	·75	·768			

As the result of this divergence the point C is broadened out into a small isoceles triangle whose apex lies on the diameter $P_0P'_0$. Owing to the symmetry about this diameter it is still possible, however, to determine the position of $P_0P'_0$ with accuracy by adopting the following method of procedure. Take any two chords PP′ and join their point of intersection with the centre of the hodograph. The point P_0, when transferred to the velocity curve, will give an approximate value of t_0, the time of passage through periastron. Select then any two values of t symmetrical on either side of t_0, and construct for these the chords PP′. Theory demands that the points of intersection of symmetrical pairs must lie on the diameter $P_0P'_0$ but at slightly different distances from the centre O. Hence, by drawing a line through these points and O, the diameter $P_0P'_0$ will be accurately determined.

To obtain a theoretically correct value for the eccentricity the following simple method may be adopted. If two points P and P′ are selected whose mean anomalies $(t - t_0) \cdot \frac{2\pi}{P}$ and $(t' - t_0)\frac{2\pi}{P}$ are $\pi/2$ and $3\pi/2$ respectively, the chord PP′ is at right angles to $P_0P'_0$ and the distance OC is :

$$OC = -k \cos v.$$

For any given eccentricity the values of v corresponding to a mean anomaly $\pi/2$ can be calculated, and the subjoined table may be constructed showing corresponding values of OC/k and e.

OC/k.	e.	OC/k.	e.	OC/k.	e.	OC/k.	e.	OC/k.	e.	OC/k.	e.
0·00	0·000	0·50	0·276	0·60	0·345	0·70	0·429	0·80	0·540	0·90	0·697
·05	·025	·51	·283	·61	·352	·71	·439	·81	·553	·91	·718
·10	·050	·52	·290	·62	·360	·72	·449	·82	·566	·92	·739
·15	·075	·53	·297	·63	·369	·73	·459	·83	·580	·93	·760
·20	·100	·54	·303	·64	·378	·74	·469	·84	·595	·94	·783
·25	·127	·55	·309	·65	·386	·75	·480	·85	·610	·95	·809
·30	·155	·56	·316	·66	·394	·76	·491	·86	·625	·96	·839
·35	·183	·57	·323	·67	·402	·77	·502	·87	·641	·97	·873
·40	·213	·58	·330	·68	·410	·78	·515	·88	·658	·98	·910
·45	·244	·59	·338	·69	·419	·79	·528	·89	·677	·99	·950
·50	·276	·60	·345	·70	·429	·80	·540	·90	·697	1·00	1·000

To give an illustration of the method for a case of large eccentricity the velocity curve was calculated for a fictitious orbit whose elements were assumed as follows :—

$t_0 = 0{\cdot}00$
$\omega = 30^\circ$
$e = 0{\cdot}500$
$k = 50$ km/sec.
$\gamma = 0$
P $= 100$ days.

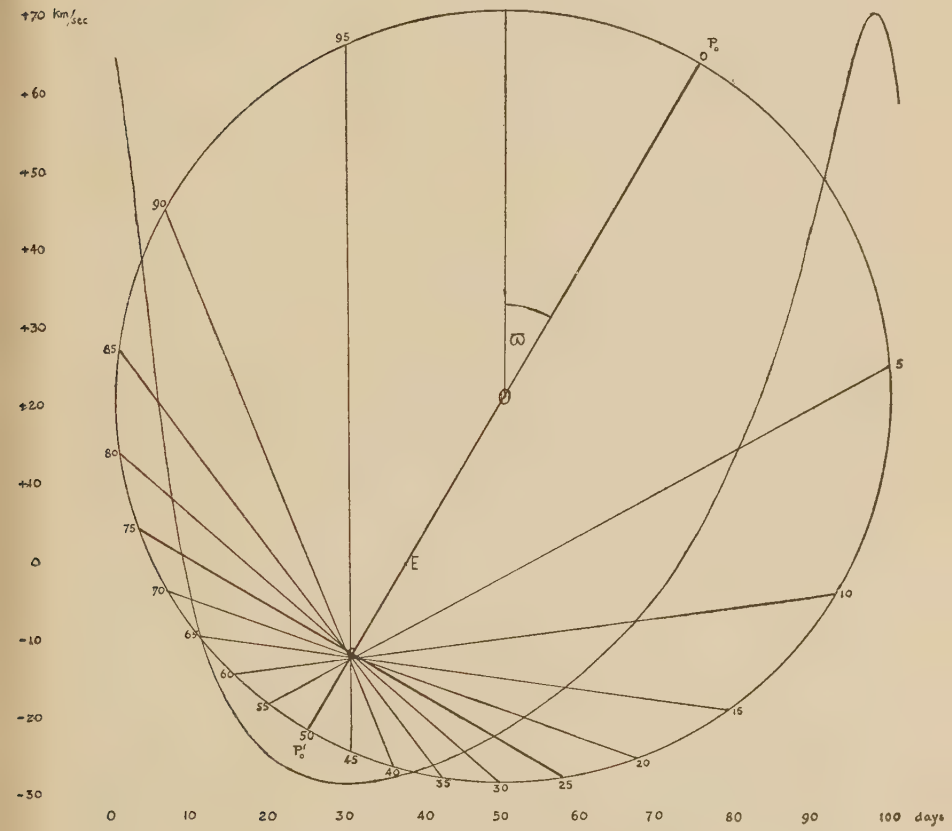


FIG. 3.

The curve and construction of the hodograph are shown in fig. 3. The graphical construction shows $\omega = 30^\circ{\cdot}0$,

$$OC/k = 38{\cdot}5/50 = 0{\cdot}770,$$

and hence from the table $e = 0{\cdot}502$.

Further, the distance $OE = 0{\cdot}502 \times 50 = 25{\cdot}1$. This places the point E exactly on the ordinate O ; hence $\gamma = 0$.

The above table can be dispensed with for eccentricities $<0{\cdot}2$ when with sufficient approximation the relation

$$OC/k = 2e$$

may be assumed.

A convenient and easily remembered empirical formula which is sufficiently accurate for eccentricities $<0{\cdot}7$ is represented by

$$\frac{OE}{OC} = \tfrac{1}{2} + \tfrac{1}{3}\left(\frac{OC}{k}\right)^3.$$

In the two examples shown in figs. 2 and 3 we find for instance :

1. $OC/k = 10/50$; $\dfrac{OE}{OC} = 0{\cdot}500$; $\dfrac{OE}{k} = e = 0{\cdot}500\dfrac{OC}{k} = 0{\cdot}100.$

2. $OC/k = 0{\cdot}770$; $\dfrac{OE}{OC} = 0{\cdot}652$; $\dfrac{OE}{k} = e = 0{\cdot}652\dfrac{OC}{k} = 0{\cdot}502.$

Increased Resolving Power in the Presence of Spherical Aberration.
By R. Kingslake, A.R.C.S., D.I.C., M.Sc.

It is fairly well known among astronomers that on certain occasions double stars have been resolved whose separations are far below the theoretical limit of resolution of the telescope. The ultimate limit of resolution of any optical instrument depends on the size of the Airy disc, both as to the central dot and also the first diffraction ring ; and if, therefore, by some means the size of the central dot can be reduced relative to the size of the first bright ring, then increased resolution of star images may be obtained.

This desirable change in the form of the Airy disc can be obtained in the presence of a considerable amount of spherical aberration, such as might arise in a telescope objective during a sudden alteration of temperature. The aim of the present work was to ascertain by experiment how much aberration is needed to produce the most desirable image from the point of view of stellar resolution, though it is important to note that such an image will not necessarily give the clearest rendering of an extended object.

As it is hardly possible to vary at will the spherical aberration of an actual telescope, it was necessary to resort to the microscope for these experiments, for in that instrument the amount of spherical aberration can be readily altered by simply changing the tube-length ; the results, of course, will apply equally well to the telescope. The chief difficulty was to find a silver-film having sufficiently fine dust-holes to act as artificial " stars." It was essential to use the most minute " stars " obtainable in order to discriminate absolutely between the true finite size of the image and the size of the Airy disc formed by diffraction. In this investigation it was very necessary to be certain that the whole appreciable size of the visible image was due to diffraction effects alone.

Another difficulty was to determine the *amount* of spherical aberration introduced by a given change in tube-length. This relation was eventually ascertained by observing the connection between the cover-thickness employed and the necessary tube-length to give an image free from spherical aberration. It is a simple matter to determine theoretically the effect of cover-thickness on spherical aberration; and hence the aberration corresponding to tube-length changes could be deduced. The objective employed was a Zeiss 8-mm. apochromat (N.A. = 0·65), used with a high-power compensating eyepiece; and with these lenses the image was as clear and well defined as it is possible to obtain.

If an observer, while studying the image of a "star" in a silver-film under a good microscope, alters the tube-length slightly and refocuses, the following changes are seen to occur. At the best tube-length and best focus there is zero spherical aberration present, and the image consists of a small round dot surrounded by a faint bright ring, the space between them appearing practically devoid of light. This is the well-known Airy disc. On now going slowly out of focus, the central dot is seen to expand into a broad white ring, swamping altogether the original narrow diffraction ring, and goes on expanding until another small point of light appears in the centre. When this dot first appears, it is *smaller* than the normal Airy disc, and it would seem that increased resolution should be obtained by using this particular focal adjustment for critical resolution with a good objective. This is, however, impracticable, as the dot is much fainter than the broad ring of light surrounding it, and the interspace between dot and ring is almost as light as the dot itself. This phenomenon is seen precisely similarly on either side of the focus, and provides a means of choosing the correct tube-length to give zero spherical aberration.

But if a small amount of spherical aberration is present, the conditions are immediately altered. The best-focused image now becomes somewhat enlarged; but more serious is the great increase in the brightness of the diffraction ring, making stellar resolution at this focal adjustment not nearly so good as with a spherically corrected instrument. On going out of focus one way, the image simply becomes diffuse and soon disappears altogether; but the other way, the central spot expands to a broad ring, and soon a minute central dot appears as before. It will now be noticed, however, that the dot is smaller, the ring wider, and the interspace less bright than in the case of a spherically corrected objective.

As more and more aberration is introduced, these conditions become more and more pronounced, until with a certain amount of aberration the central dot is seen as a clear, sharply defined point of light, in the centre of a pitch-black circle about three times its size, the whole being surrounded by a sharply defined broad bright ring. The whole of the light appears to be concentrated in the ring and the dot, leaving none in the interspace whatever. This is the optimum amount of spherical aberration for increased resolution, because, if still more aberration is introduced, a new faint ring of light appears in the interspace, at once spoiling the observer's chance of detecting another

close image there, as he would be trying to do in resolving a double star. The size of the central dot in this case is less than half that of the normal Airy disc.

With the Zeiss microscope objective used for this work, the cover-thickness being 0·1 mm., it was found that a change in the tube-length of 10 mm. from the position of zero spherical aberration (which was at T.L. = 19·64 cm.) gave this optimum amount of aberration; and it was now necessary to determine how much spherical aberration had been introduced, in terms of the Rayleigh limit.

As has been already mentioned, this was done by using a series of silver films under cover-glasses of successively increasing thickness, in each case determining experimentally that tube-length which gave an Airy disc appearing similar when expanded on either side of the focus; and a graph was drawn connecting cover-thickness and tube-length.

We will now refer to Professor Conrady's expression * for the optical path difference W arising at any spherical surface, where n, n' are the refractive indices before and after the surface; ϕ, ϕ' the angles of incidence and refraction; β, β' the convergence angles of the incident and refracted rays; $\alpha = \phi + \beta = \phi' + \beta'$; and y is the incidence height.

$$W = \frac{n'y \sin \phi'}{2 \cos \frac{\alpha}{2}} \left\{ \frac{1}{\cos \frac{\beta'}{2} \cos \frac{\phi'}{2}} - \frac{1}{\cos \frac{\beta}{2} \cos \frac{\phi}{2}} \right\} \quad . \quad . \quad (1)$$

This may be modified for refraction at a plane surface such as the surfaces of our cover-glass, by writing $\phi = -\beta$; $\phi' = -\beta'$; $\alpha = 0$. Hence for a *plane* surface:

$$W = \frac{y}{2}(n' \sin \beta') \left\{ \frac{1}{\cos^2 \frac{\beta}{2}} - \frac{1}{\cos^2 \frac{\beta'}{2}} \right\} \quad . \quad . \quad (2)$$

To discover the whole effect of a cover-glass of thickness t and refractive index n, consider fig. 1. The "star" under observation is supposed to be in contact with the cover-glass at the point O, and the line OPQ represents the path of the extreme ray which enters the microscope objective.

There will be no path difference arising at the first surface of the cover-glass, because in the equations for W the incidence-height $y = 0$. But at the second surface we have $y = t \tan \beta$, and the path difference arising is therefore, by (2),

$$W = \frac{t \tan \beta}{2} \sin \beta' \left\{ \frac{1}{\cos^2 \frac{\beta}{2}} - \frac{1}{\cos^2 \frac{\beta'}{2}} \right\} \quad . \quad . \quad (3)$$

The numerical aperture of the Zeiss 8 mm. objective used for these experiments was $\sin \beta' = 0{\cdot}65$, giving the value of β' for the extreme

* *M.N.*, **65** (1905), 602, Equation 4.

ray as 40° 32′ 30″. The angle β can now be computed by the ordinary law of refraction, namely, $n \sin \beta = \sin \beta'$; but, in order to apply this law, some numerical value for n must be chosen. Actually several arbitrary values of n were tried and the corresponding values of W calculated by equation (3), but it was found that by some fortuitous circumstance the precise numerical value of the refractive index has little effect on the path difference W. This is because it so happens that W reaches a maximum (in the mathematical sense) for $n = 1{\cdot}54$, and hardly varies appreciably over the whole range from $n = 1{\cdot}5$ to

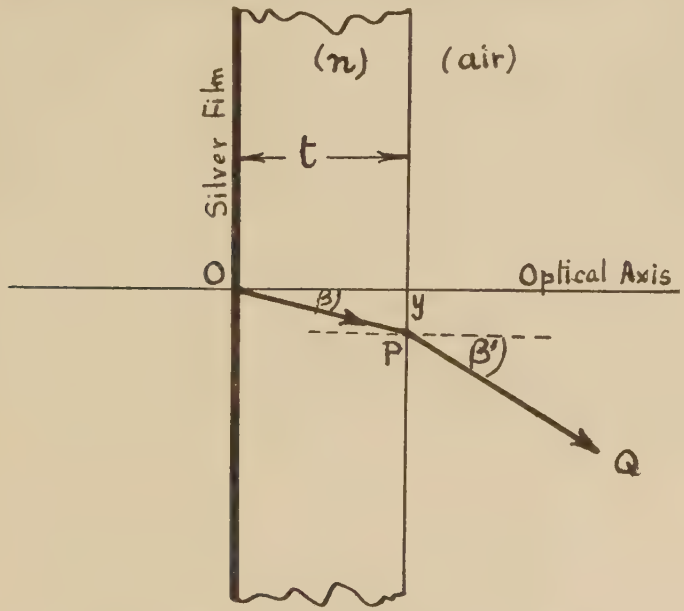


Fig. 1.—Path of extreme ray through the cover-glass.

$n = 1{\cdot}6$. Hence, for this calculation, it was decided to adopt the value 1·54 for the refractive index n.

This gave $\beta = 24° 57′ 56″$, and finally we have $W = 0{\cdot}01322t$. It is, however, customary and more convenient to express W in terms of the mean wave-length of visible light (say 0·0005 mm.), and to express the cover-thickness t in mm. In these units we have

$$W_{(\lambda)} = 26{\cdot}44t_{(\text{mm})} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

A series of silver-films were deposited on cover-glasses of thicknesses 0·09, 0·10, 0·11, . . . 0·22 mm. respectively; and in each case the required tube-length to give zero spherical aberration was found by trial. In this way a curve was obtained connecting changes in tube-length with the corresponding changes in spherical aberration expressed as a path-difference by equation (4). From this graph, which was very nearly a straight line, it was seen that the change in tube-length

from 19·64 cm. to 18·64 cm. had introduced a spherical aberration for the extreme ray of 1·54 wave-lengths. Now the formula (1) above gives path-differences relative to the *marginal* image-point; but it can be shown that, relative to the *best* focus, the value of W is only one-quarter of this amount. Hence Lord Rayleigh's limit of $\frac{1}{4}\lambda$ for the tolerable path-difference at the best focus becomes 1λ for the marginal focus; and thus we see from these experiments that the maximum visual resolving power for close double stars will be obtained when there is spherical aberration present to the extent of $1\frac{1}{2}$ times the Rayleigh limit.

This result is very interesting, as it shows that a lens with a decidedly undesirable amount of spherical aberration can actually give better resolution on close double stars than a perfectly corrected objective. It should, however, be carefully noted that such a lens could not give a clear image of an extended object. The resolution of close double stars must occur *within* the broad bright diffraction ring which surrounds the small central dot of light, and, owing to the large amount of diffused light, an extended object such as a planet or the Moon would appear so fogged that clear definition would be impossible.

Since some types of telescope objective when changing temperature may easily acquire as much spherical aberration as $1\frac{1}{2}$ times the Rayleigh limit, it seems probable that those cases in which an observer has resolved stars considerably closer than the theoretical limit of resolution of his instrument can be explained on the basis of this phenomenon.

I am greatly indebted to Professor Conrady for suggesting this study and for kindly lending me his Zeiss apochromatic microscope objective.

Observations of R T Cygni (Ch. 7085) in 1922–27.

By A. N. Brown, M.A.

The observations of this long-period variable recorded below number 256, and are in continuation of those published in *Monthly Notices*, **82**, 405–410. From light-curves drawn from them the following ten maxima and ten minima have been determined:—

	Maxima.				Minima.			
No.	Date.	Julian Date.	Ded. Mag.	No.	Date.	Julian Date.	Ded. Mag.
28	1922 Aug. 1	3268	6·6	28	1922 May 12	3187	10·9
29	1923 Feb. 10	3461	7·9	29	1922 Dec. 1	3390	11·3
30	1923 Aug. 31	3663	7·4	30	1923 May 30	3570	10·9
31	1924 Mar. 9	3854	7·7	31	1923 Dec. 12	3766	10·9
32	1924 Sept. 19	4048	7·4	32	1924 June 13	3950	11·2
33	1925 Apr. 3	4244	7·6	33	1925 Jan. 3	4154	10·9
34	1925 Oct. 16	4440	7·4	34	1925 July 15	4347	11·2
35	1926 Apr. 3	4609	6·3	35	1926 Jan. 16	4532	10·9
36	1926 Oct. 13	4802	7·9	36	1926 July 28	4725	11·3
37	1927 Apr. 23	4994	7·4	37	1927 Jan. 29	4910	11·3

The number given on the left of each date in the above table shows the number of maxima and minima which have occurred since my first observations of this variable in 1908 which appear in *Monthly Notices*, **69**, 585–588. It is noticeable that, as on previous occasions when the fall to minimum has been exceptionally protracted, the following maximum is a faint one. Thus the interval between maximum No. 28 and minimum No. 29 is 122 days, the mean interval being 105 days; and again 116 days elapsed between maximum 35 and minimum 36. In each case the subsequent maxima, Nos. 29 and 36, are 7·9 mag. as compared with 7·1 mag., which is the mean maximum light since these observations began.

The mean period, using minima as well as maxima, derived from the present series of observations is 190·4 days.

The observations have been arranged in the same way as those of V Cassiopeiæ which appeared last year in *Monthly Notices*, **86**, 631–636, and it therefore seems unnecessary to repeat the explanations then given. As previously, the telescopes used were 5-inch and 3-inch refractors with powers varying from 25 to 200.

The Observations.

Date.	Julian Date.	Comparisons.	Ded. Mag.
1922			
May 2	3177	44+1	10·9
6	3181	43(3)v(1)44	10·9
14	3189	43(3)v(1)44	10·9
19	3194	43(3)v(1)44	10·9
21	3196	43(3)v(1)44	10·9
23	3198	43(3)v(1)44	10·9
27	3202	43(3)v(2)44	10·8
June 1	3207	41(2)v(1)43	10·5
4	3210	41(1)v(1)43	10·5
6	3212	41(1)v(2)43	10·5
July 18	3254	3(2)v(1)4	7·3
20	3256	3(2)v(1)4	7·3
28	3264	2−1, 3+1	6·6
30	3266	2−1, 3+1	6·6
Aug. 7	3274	3(1)v(2)4	7·0
15	3282	3(1)v(2)4	7·0
23	3290	3(1)v(1)4	7·2
25	3292	3(3)v(2)4	7·3
31	3298	3(2)v(1)4	7·3
Sept. 9	3307	=4, 5+1·5	7·6
20	3318	7(2)v(1)10	8·3
24	3322	10−1	8·5
29	3327	14(1)v(3)19	8·8
Oct. 6	3334	=19	9·3
Oct. 12	3340	19(1)v(1)22	9·4
14	3342	22+1·5	9·4
22	3350	23(2)v(1)25	9·6
27	3355	41(2)v(1)43	10·5
Nov. 1	3360	43(3)v(1)44	10·9
13	3372	44−1	11·1
15	3374	44−1	11·1
18	3377	44−2	11·2
24	3383	44−3	11·3
Dec. 8	3397	44−3	11·3
18	3407	43(3)v(1)44	10·9
23	3412	43(2)v(1)44	10·9
26	3415	43(1)v(1)44	10·8
1923			
Jan. 4	3424	25−5, 41+2	10·2
8	3428	25(5)v(2)41	10·2
11	3431	25(3)v(2)41	10·1
14	3434	23(1)v(1)25	9·6
20	3440	14(5)v(2)19	9·1
23	3443	14(1)v(1)19	9·0
Feb. 4	3455	5(2)v(1)7	7·9
11	3462	5−1, 7+1	7·9
Mar. 6	3485	5(3)v(1)7	8·0
17	3496	10(1)v(2)14	8·5

The Observations—continued.

Date.	Julian Date.	Comparisons.	Ded. Mag.
1923			
Apr. 9	3519	19(2)v(1)22	9·4
13	3523	23−1, =25	9·7
26	3536	=41	10·4
May 5	3545	43(3)v(1)44	10·9
17	3557	44+0·5	11·0
21	3561	43(3)v(1)44	10·9
25	3565	43(4)v(1)44	10·9
June 11	3582	43(5)v(2)44	10·9
July 4	3605	30(3)v(1)41	10·3
8	3609	30(3)v(1)41	10·3
20	3621	14(5)v(1)19	9·2
24	3625	14−4·5, 19+2	9·1
28	3629	14−4, 19+2	9·1
Aug. 3	3635	14−1	8 8
8	3640	=7	8·0
11	3643	5(3)v(1)7	8·0
13	3645	5+1	7·7
18	3650	=4, 5+2	7·6
24	3656	3−7, 4+3	7·4
30	3662	3−7, 4+3	7·4
Sept. 5	3668	3(4)v(1)4	7·4
8	3671	3(7)v(1)4	7·5
11	3674	3(3)v(2)4	7·3
23	3686	5(1)v(1)7	7·9
Oct. 4	3697	7(2)v(1)10	8·3
14	3707	14−2, 19+5	8·8
22	3715	=19	9·3
28	3721	19(3)v(2)22	9·4
Nov. 3	3727	22(2)v(1)25	9·6
7	3731	41+3	10·1
10	3734	=41	10·4
18	3742	43(1)v(3)44	10·7
Dec. 2	3756	43(3)v(1)44	10·9
15	3769	44+1	10·9
23	3777	43(3)v(1)44	10·9
1924			
Jan. 2	3787	43(1)v(1)44	10·8
6	3791	41(2)v(1)43	10·5
9	3794	41(2)v(5)43, 41−1	10·5
14	3799	25(1)v(3)30	9·8
20	3805	14(5)v(1)19	9·2

Date.	Julian Date.	Comparisons.	Ded. Mag.
1924			
Jan. 25	3810	14(1)v(4)19	8·8
Feb. 1	3817	=10	8·4
4	3820	7(1)v(1)10	8·2
16	3832	5(1)v(1)7	7·9
24	3840	4(3)v(2)5	7·7
Mar. 2	3847	4(4)v(1)5	7·8
4	3849	4(3)v(1)5, 5+1·5	7·7
8	3853	4(1)v(1)5	7·7
29	3874	5(2)v(1)7	7·9
Apr. 1	3877	5(2)v(1)7	7·9
4	3880	5−2, =7	8·0
7	3883	5(2)v(1)7	7·9
20	3896	14(2)v(3)19	8·9
May 1	3907	14−6, 19+0·5	9·3
5	3911	19(1)v(1)22	9·4
16	3922	41(2)v(1)43	10·5
27	3933	43(3)v(1)44	10·9
June 13	3950	44−2	11·2
20	3957	44−1	11·1
July 10	3977	=43	10·6
13	3980	41(2)v(1)43	10·5
19	3986	30(2)v(1)41	10·3
24	3991	22−3, 23−3, 30+1	9·8
29	3996	=25	9·7
Aug. 2	4000	19(2)v(1)22, 23+1	9·4
7	4005	14(7)v(1)19	9·2
12	4010	14(3)v(1)19	9·1
25	4023	7(1)v(2)10	8·1
31	4029	4(3)v(2)5	7·7
Sept. 11	4040	3(4)v(1)4	7·4
14	4043	3−7, 4+3	7·4
22	4051	3(7)v(3)4	7·3
27	4056	3(4)v(1)4	7·4
Oct. 12	4071	4(3)v(1)5	7·8
19	4078	=7	8·0
23	4082	7(3)v(1)10	8·3
Nov. 3	4093	19+1·5, 22+3	9·2
16	4106	30−3·5, 41+1	10·3
27	4117	41(2)v(1)43	10·5
Dec. 5	4125	43(2)v(3)44	10·8
14	4134	43(1)v(1)44	10·8

The Observations—continued.

Date.	Julian Date.	Comparisons.	Ded. Mag.
1924			
Dec. 26	4146	43(3)v(1)44	10·9
30	4150	43(3)v(1)44	10·9
1925			
Jan. 4	4155	43(3)v(1)44	10·9
6	4157	43(3)v(1)44	10·9
15	4166	43(3)v(1)44	10·9
22	4173	43+1	10·5
Feb. 1	4183	25−2, =30	10·0
6	4188	22+1, 23+1	9·4
16	4198	14(2)v(1)19	9·1
22	4204	14−0·5	8·7
28	4210	=7	8·0
Mar. 8	4218	4(1)v(1)5	7·7
12	4222	4−1, 5+1, 7+3	7·7
18	4228	4(1)v(1)5	7·7
21	4231	4(1)v(1)5, 7+3	7·7
Apr. 1	4242	5+2, 7+4	7·6
3	4244	=4, 5+2	7·6
8	4249	4(1)v(1)5, 7+4	7·7
12	4253	4(1)v(1)5	7·7
21	4262	4(2)v(1)5, 7+3	7·7
May 12	4283	14−2, 19+5	8·8
20	4291	14(5)v(1)19, 22+2	9·3
25	4296	23(2)v(1)25, 22−1	9·6
28	4299	22(3)v(1)30	9·9
June 1	4303	41+1·5	10·3
8	4310	43+1	10·5
11	4313	43+1	10·5
17	4319	43(3)v(1)44	10·9
23	4325	=44	11·0
29	4331	44−1	11·1
July 10	4342	44−2	11·2
14	4346	44−2	11·2
18	4350	44−2	11·2
Aug. 2	4365	43(2)v(1)44	10·9
7	4370	41(3)v(1)43	10·6
11	4374	41(1)v(3)43	10·5
16	4379	25(4)v(1)30	9·9
25	4388	14(5)v(1)19	9·2
28	4391	14(3)v(2)19	9·0
Sept. 4	4398	7(1)v(2)10	8·1
1925			
Sept. 10	4404	5(1)v(1)7	7·9
14	4408	=4, 5+2	7·6
21	4415	3(9)v(1)4	7·5
26	4420	3(9)v(1)4	7·5
Oct. 8	4432	3(4)v(1)4	7·4
14	4438	3(4)v(1)4	7·4
18	4442	3−7, 4+3	7·4
24	4448	3(8)v(2)4	7·4
27	4451	=4, 5+3	7·6
Nov. 3	4458	5(1)v(2)7	7·9
8	4463	7(1)v(2)10	8·1
15	4470	7(1)v(1)10	8·2
25	4480	14(1)v(3)19	8·8
28	4483	14(4)v(1)19, 22+3	9·2
Dec. 2	4487	14(5)v(1)19	9·2
5	4490	14(7)v(1)19, 22+2	9·3
11	4496	19−4, 22+1, 23+1, 25+3	9·5
17	4502	22(3)v(1)25, 23−1	9·6
1926			
Jan. 3	4519	43(1)v(3)44	10·7
6	4522	43(1)v(1)44	10·8
11	4527	43(2)v(1)44	10·9
16	4532	43(3)v(1)44	10·9
19	4535	43(3)v(1)44	10·9
Feb. 13	4560	22+1·5	9·4
25	4572	14(1)v(1)22, 14−4·5	9·1
Mar. 9	4584	5(1)v(1)7	7·9
13	4588	3(4)v(1)4	7·4
15	4590	3(1)v(1)4	7·2
21	4596	=2, 3+2	6·5
26	4601	2+2	6·3
30	4605	1−1, 2+2	6·3
Apr. 5	4611	1(1)v(3)2	6·3
9	4615	1−1, 2+2	6·3
15	4621	1(1)v(1)2	6·4
17	4623	1(2)v(1)2	6·4
May 3	4639	2(3)v(2)3	6·6
9	4645	2(2)v(1)3	6·6
13	4649	3(4)v(1)4	7·4

The Observations—continued.

Date. 1926	Julian Date.	Comparisons.	Ded. Mag.
May 15	4651	4+2	7·4
20	4656	5(2)v(1)7	7·9
31	4667	10(3)v(1)14	8·6
June 5	4672	19+2, 22+4, 23+4	9·1
15	4682	41+3	10·1
22	4689	41(1)v(2)43	10·5
26	4693	=43	10·6
July 13	4710	44−1	11·1
15	4712	44−2	11·2
21	4718	44−3	11·3
30	4727	44−3	11·3
Aug. 1	4729	44−3	11·3
6	4734	44−2·5	11·3
14	4742	43(3)v(1)44	10·9
26	4754	41+1	10·3
Sept. 4	4763	25(2)v(1)30	9·9
10	4769	22+2, 23+2	9·3
13	4772	14(3)v(1)22	9·3
27	4786	10(3)v(1)14	8·6
Oct. 2	4791	7(3)v(1)10	8·3
8	4797	5(2)v(1)7	7·9
18	4807	5(1)v(1)7	7·9
23	4812	7(1)v(1)10	8·2
26	4815	7(3)v(1)10	8·3
30	4819	7(3)v(1)10	8·3
Nov. 5	4825	7(3)v(1)10	8·3
12	4832	10(1)v(4)14	8·4
20	4840	14(3)v(1)19	9·1
Dec. 2	4852	22−1, 23−1, 25+1	9·6
8	4858	23(2)v(1)25	9·6
15	4865	41(2)v(1)43	10·5
27	4877	43(3)v(1)44	10·9
29	4879	43(5)v(1)44	10·9
1927			
Jan. 4	4885	44+0·5	11·0
21	4902	44−3	11·3
25	4906	44−3	11·3
Feb. 2	4914	44−3	11·3
4	4916	44−3	11·3
24	4936	41(3)v(1)43	10·6
Mar. 23	4963	25(2)v(1)30	9·9
30	4970	=7	8·0
Apr. 15	4986	3(8)v(2)4	7·4
19	4990	3(7)v(1)4	7·5
26	4997	3(8)v(2)4	7·4
30	5001	3(4)v(1)4	7·4
May 4	3500	4+1	7·5
7	5008	=4	7·6
11	5012	4(1)v(1)5	7·7
28	5029	=14	8·7
June 3	5035	14(1)v(3)19	8·8

The Comparison Stars Used.

Hagen's Number.	Magnitude. H.P.	Magnitude. Hagen.
1	6·24	..
2	6·50	..
3	6·67	..
4	7·64	..
5	..	7·8
7	..	8·0
10	8·39	..
14	8·67	..
19	..	9·3
22	..	9·5
23	..	9·5
25	..	9·7
30	..	10·0
41	..	10·4
43	..	10·6
44	..	11·0

Brackenhurst,
Bucklebury Common, Berks:
1927 *June* 7.

Z Aquarii (234716, *Ch.* 8562). By P. M. Ryves.

1. The variability of this star, which is B.D. − 16° 6379, was discovered by Mrs. Fleming in 1896. Hartwig deduced a period of 216 days and epoch for maximum J.D. 2415221, and these elements and a range from 8·0 to 9·2 are given in the *Second Catalogue* of Harvard, but the note is added : " No regular period could be derived from measures of Harvard photographs, extending from January 21, 1888, to October 18, 1895."

In spite of discrepancies, Hartwig's elements, as above, were used for the predictions appearing in *Kat. und Eph.* until comparatively recently, but in the 1924 issue the period is marked doubtfully " 136 ? " and no predictions are made.

2. In view of the continued uncertainty as to the period and type of variation of this star, regular observations, which had been projected some years previously, were commenced here in 1924.

The variable was identified with the help of a photograph (D 11703) kindly supplied by the late Professor E. C. Pickering, and the Harvard magnitudes of two stars shown on the photograph were used in adjusting the light scale of grades to the magnitude scale of H.C.O. This has been done in a provisional manner, the data being insufficient for a complete investigation of the sequence, and some of the discordancies in the observations may be due to the values provisionally assigned to the standards.

3. Sixty-two visual observations were made during the last three seasons, June to February. In the first season two minima were well observed and in the last two maxima. In 1925 the observations obtained were few, but they indicate that a broad maximum took place in November, whilst a single observation on June 29 shows that Z Aquarii was at or near maximum on that date.

The majority of the observations have been made with a 3-inch refractor and powers of 20 and 30, but a 5-inch was used when the intrinsic faintness of the variable or the presence of moonlight, haze, etc., made greater light-grasp an advantage. One observation made with a different instrumental combination has been discarded.

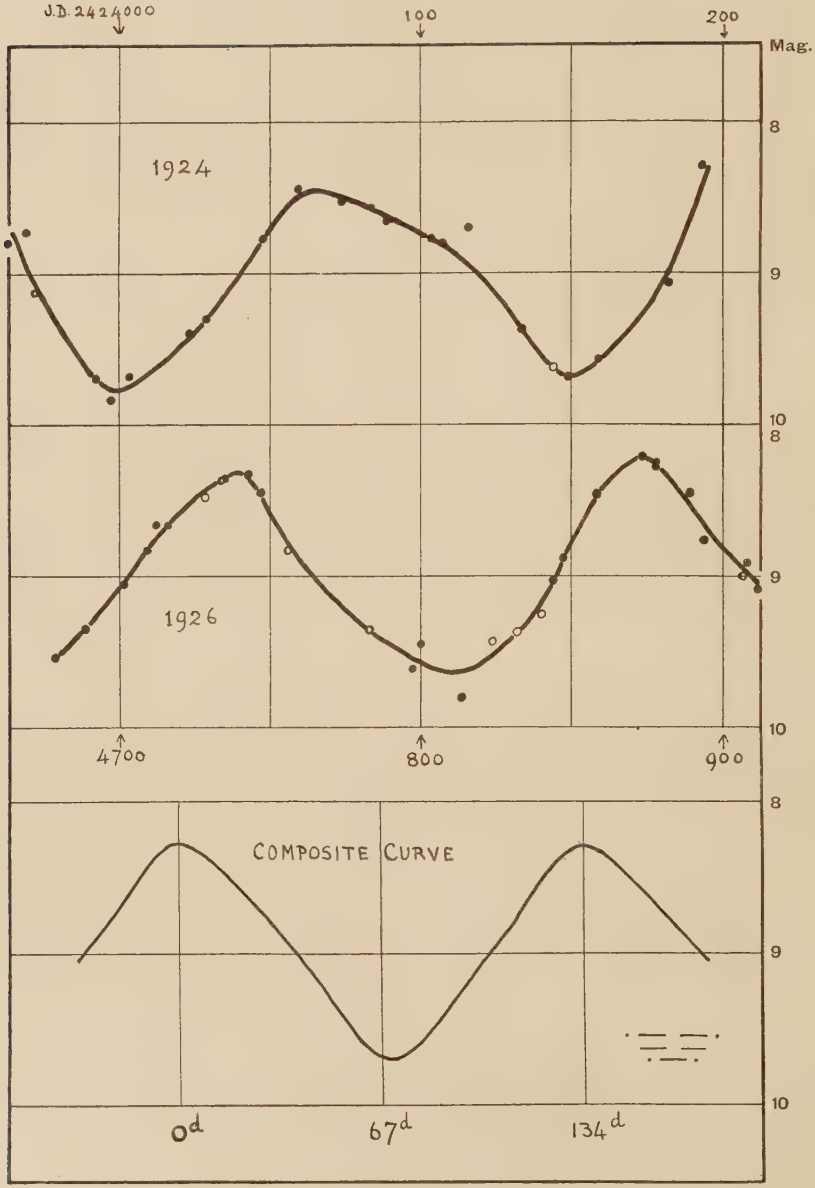
4. The general character of the variation has been inferred from the above material.

The observed range (extreme) is from magnitude 7·2 to 9·7. A period of 134 days is indicated, and the curve is nearly symmetrical, $M - m$ being 63 days, roughly. The shape of the curve differs from cycle to cycle, and, apart from the small amplitude, the variation appears to be similar to that of typical long-period variables of the same spectral type, M*d*. The following elements have been deduced :—

$$\text{Max.} = \text{J.D. } 2424064 + 134E.$$

According to these, the next maxima should take place 1927 October 12 and 1928 January 24.

Carrying the computation backwards, a maximum would fall very



Z Aquarii: Light-Curves.

Scale: 1 mag. = 50 days = 2 cms.
The circles represent second-class observations.

near Hartwig's original epoch. The period of 216 days found by him was, no doubt, the result of assuming that two instead of three cycles (or multiples) had elapsed between isolated maxima recorded in short observing seasons.

The above conclusions are the fruit of a preliminary examination of a short series of observations and are subject to revision. The small amplitude of the variation and rather red colour of the star, combined with its unfavourable position in the sky, render observation somewhat difficult.

Zaragoza, Spain:
1927 June.

Note on the Shadow-Bands. By P. M. Ryves.

After the eclipse of 1905, total in Spain, I became convinced that the shadow-bands were the result of light from a very small source, a small segment of the Sun's disc, revealing the irregularities of density in the atmosphere, in a manner similar to that in which the irregularities in the glass of a window-pane are revealed by the light of the planet Venus, producing a mottled pattern on the wall; the linear or banded character of the shadows seen during an eclipse being due to the fact that in this case the source of light, though very small in one direction, is of appreciable length in the direction of its major axis. Light from the ends of the segment will "shine round" and so not cast shadows of linear objects, or inequalities, lying in planes much inclined to that in which the major axis is situated, but those linear objects which lie in the plane of the major axis will cast shadows as though the source were a point. Masses of air at different temperatures generally exist, especially near the ground in sunny weather, and the boundary lines separating them mark the direction of the refraction-shadows revealed by light from a very small and bright source.

Some observations and experiments with artificial lights supported this view. A naked arc-light at a distance of about 200 metres produced shadow mottling on many occasions and a search-light distant about 2 kilometres showed a very strong mottling. A fair imitation of shadow-bands was produced in the physical laboratory of the University by means of electric lights and a chemical combustion furnace. Faint atmospheric shadows were also seen cast by Venus when near W. elongation in 1910.

On reading up the subject I found that the light proceeding from a very small portion of the Sun's disc had been suggested as the cause of the shadow-bands some years previously by Professor Todd in his little book on eclipses, and this is also favoured by Mr. Maunder in his volume *The Indian Eclipse*; but no explanation of the linear or banded character of the shadows was advanced.

It follows from the explanation given above that the shadow-bands must always lie in a plane tangent to the Moon's disc at the points of 2nd and 3rd contact or the position angle of greatest phase in a partial

eclipse. Just outside the zone of totality the bands will be parallel to the track; on the central line they will be at right angles to the track both before and after totality, whilst at intermediate points they will occupy different directions before and after the total phase.

I should like to ask those who make or have made observations to be careful in publishing them to give the *orientation* (compass bearing of the direction of the bands themselves), as distinguished from the direction of *motion*, which latter is fundamentally of less import, being dependent on local air currents.

Zaragoza, Spain:
1927 *June.*

Determination of the Longitude of Pulkovo Observatory in 1925.

By J. Beljajeff and N. Dneprovsky.

(*Communicated by the Secretaries.*)

No previous direct determination of difference of longitude between Pulkovo and Greenwich exists, values having been obtained indirectly as follows:

λ = 2^h 1^m $18^s.67$	in 1843–44	(through Altona, by transport of chronometers).
„ „ $18^s.55$	„ 1901 and 1903	(through Potsdam, by line).
„ „ $18^s.64$	„ 1914	(through Paris, by wireless).

The longitude $18^s.570$, deduced by Professor T. Albrecht from the general solution of longitudes of European Observatories, ought to be considered the most probable.

This made the Observatory of Pulkovo strive for a direct determination of longitude by means of wireless. The organisation of this work found sympathy and support in Sir Frank Dyson, Astronomer Royal, to whom chiefly we owe the successful completion of the work. We are glad of the opportunity to express here our deepest gratitude to Sir Frank Dyson, Dr. Jackson, Mr. Bowyer, and the staff of the Observatory of Greenwich. We ought to mention gratefully also the valuable assistance rendered by Professor F. Blumbach.

The observations were begun on 1925 August 22 and closed on November 15. Mr. J. Beljajeff worked in Greenwich for the first half of the time, while Mr. N. Dneprovsky was doing his part in Pulkovo. At the end of September the observers with all their instruments and receiving apparatus changed places. For the observations in Greenwich the Transit Pavilion and the clock Shortt No. 3 were kindly placed at the disposal of the observers, and a special aerial was erected. In Pulkovo the observations were carried out in the "Southern Observatory" with clock Riefler No. 451.

For the astronomical observations two Bamberg transit instruments of the same type supplied with self-registering micrometers were used. The complete determination of the clock-correction comprised the registration of the transits of 18 to 20 circumzenithal (clock) and 4 to 6 polar

stars. As basis for the places of stars the Pulkovo Catalogue of right ascensions 1915·0 was taken, the declinations of the clock-stars having been selected within the limits 40° to 62° and the declinations of the polar stars from 83° to 86°. The polar stars used alternated as far as possible in different culminations. The registration of the times of transits and of the wireless signals was done with the same pen of a Hipp chronograph, the second pen being actuated by the signals of the clock through polarised relays. For the automatic registration of wireless signals two uniform receiving sets were built by Messrs. L. Termen and A. Konstantinoff. These receiving sets are composed of a receiver, two amplifiers, one of radio and one of audio frequency oscillations and two supplementary valves in parallel connection, the anode circuit being in the relay winding.

The reception of undamped oscillations is effected by the method of beats, using for that purpose a separate heterodyne with inductive coupling. The elaboration of electric schemes and the investigation of the lag of the relays were done by Professor P. Asboukin by means of an oscillograph.

The following table gives the summary of observations, obtained during the time of the expedition :

Place.	Obs.	Number of Clock-stars.	Number of Polar Stars.	Number of Days of Observations.
Greenwich	Bl.	253	85	19
Pulkovo	Dn.	317	107	18
Pulkovo	Bl.	238	75	17
Greenwich	Dn.	491	119	29

The preliminary reduction of the observations and of the reception of signals was made partly in Pulkovo, but chiefly in the Computing Department of the Observatory in Leningrad.

As the instruments were reversed for each star, differences of azimuth in the E.-W. and W.-E. position of the eyepiece were formed. These differences, as well as the individual corrections of R.A. of polar stars proved practically zero. The comparison of azimuths, deduced from the different culminations of polar stars, gave a certain systematic difference.

Individual corrections to the adopted R.A.'s have been obtained for all clock-stars observed at least ten times, and they have been used in the following discussion. A good agreement in these corrections by the two observers confirms the high qualities of the instruments.

The precision of the determination of final azimuths and clock-corrections for each night, taking into consideration the hourly changes of their rates, is evident from the following figures :

Place.	Obs.	Prob. Error of Azimuth.	Prob. Error of Clock-corr.
Greenwich	Bl.	± $0^s·008$	± $0^s·005$
Pulkovo	Dn.	13	07
Pulkovo	Bl.	08	06
Greenwich	Dn.	10	05

The programme for the reception of wireless signals consisted of automatic registration of Bordeaux at 20^h and Nauen at 0^h. In addition, soon after the beginning of the work it was found desirable to register in Greenwich the wireless signals from Nauen at 12^h.

The summary of reception of wireless signals is given below :

Place.	Time.	Obs.	Nauen 12^h.	Bordeaux 20^h.	Nauen 0^h.
Greenwich	Aug. 19–Sept. 17	Bl.	25	32	20
Pulkovo	Aug. 17–Sept. 13	Dn.	..	18	16
Pulkovo	Oct. 5–Nov. 16	Bl.	40	43	31
Greenwich	Oct. 2–Nov. 14	Dn.	30	35	20

The automatic registration of wireless signals was elaborated as follows : for each reception of the same station forty common signals were selected—ten signals at the beginning, ten at the end, and twenty signals in the middle of the transmission. The half-sums of times of signals, symmetrical with relation to the middle signal, gave the moment of mid-transmission by the clock of the observer. The probable error of the mean of twenty double signals deduced from the deviations of individual pairs proved equal to $\pm 0^s.001$ with both observers.

In the following table are given the longitudes from individual receptions, which served for the deduction of final results.

The values of λ are the longitudes deduced from the receptions ; the intervals between these and the determinations of clock-corrections obtained by both observers were never longer than $\pm 48^h$. The signs (1) and (2) denote the longitudes from such receptions when the above-mentioned intervals were no longer than ± 6 and ± 12 hours.

First Series.

Date.	Gr. Mean Time.	λ.	Date.	Gr. Mean Time.	λ.
	h	h m s		h	h m s
Aug. 24	0	2 1 18·560	Sept. 3	20	2 1 18·605 (1)
24	20	·584	4	0	·587 (1)
25	20	·586 (1)	4	20	·569 (1)
26	0	·581 (1)	5	0	·567 (1)
26	20	·603 (1)	5	20	·593
27	20	·591	6	20	·615
28	20	·578	7	20	·633
29	0	·574	9	20	·624 (1)
30	0	·593	10	0	·604 (1)
31	20	·609 (1)	11	20	·582 (1)
Sept. 1	0	·589 (1)	12	20	·631 (1)
1	20	·604 (1)	13	0	·619 (1)
2	12	·602 (2)	13	20	·592 (1)
2	20	·622	14	0	·568 (1)

Second Series.

Date.	Gr. Mean Time. h	λ. h m s	Date.	Gr. Mean Time. h	λ. h m s
Oct. 6	20	2 1 18·611	Oct. 19	0	2 1 18·601
8	12	·602 (2)	20	12	·594
8	20	·585 (1)	20	20	·594
9	12	·565	31	12	·554 (2)
10	0	·593	31	20	·562 (1)
10	12	·573 (2)	Nov. 1	0	·567 (1)
10	20	·546 (1)	1	12	·550 (2)
11	0	·537 (1)	1	20	·564
11	20	·594 (1)	2	0	·556
12	0	·553 (1)	2	12	·532 (2)
12	20	·537	3	0	·532 (2)
13	12	·538	8	20	·558
13	20	·553	9	20	·565 (1)
16	12	·605	11	20	·571
17	0	·581	12	12	·563 (1)
17	12	·616 (2)	12	20	·573 (1)
18	0	·551 (1)	13	12	·566 (2)
18	20	·597	14	12	·592

The means taken separately from $\lambda^{(1)}$ and $\lambda^{(1)}$ and $\lambda^{(2)}$ are given in the following table :

Groups.	*First Series.* The Mean λ.	Prob. Error.	No. of λ.	*Second Series.* The Mean λ.	Prob. Error.	No. of λ.
$\lambda^{(1)}$	$18^s.595$	$\pm 0^s.003$	17	$18^s.563$	$\pm 0^s.003$	11
$\lambda^{(1)}$ and $\lambda^{(2)}$	·596	3	18	·564	3	19
All λ	·595	2	28	·568	3	36

As a final result for each series the mean from all values of λ was taken with the weights : 1 for $^{(1)}$ and $^{(2)}$, and $\frac{1}{2}$ for other longitudes.

Further, the results were reduced to the centres of both Observatories by the application of a correction of $-0^s.010$; also, in order to reduce them to uniform conditions of electric circuits of both receiving installations, corrections of $-0^s.008$ for the first series and $+0^s.008$ for the second were introduced.* The last reduction does not influence the final result. Thus we get from the 1st and 2nd series the following longitudes of the centre of Pulkovo Observatory in relation to the meridian transit circle of Greenwich :

1st Series $\lambda = 2^h\ 1^m\ 18^s.577 \pm 0^s.0024$ (prob. error).
2nd Series $\lambda = 2^h\ 1^m\ 18^s.566 \pm 0^s.0026$,, ,,
Mean $\lambda = 2^h\ 1^m\ 18^s.572 \pm 0^s.0018$,

which is accepted as final.

* The relay of Mr. Beljajeff's receiver (No. 1) was shunted by a condenser of 0·5 microfarads, that of Mr. Dneprovsky's receiver (No. 2) by a condenser of 2·0 microfarads. The investigation by means of an oscillograph showed that according to this difference of capacity the lag of the receiver No. 2 was greater than that of the receiver No. 1 by $0^s.008$.

Pulkovo Observatory :
1927 *May* 7.

Observations occasionnelles faites à l'Institut Astronomique de l'Université Charles, Prague-Smichov. Par Emile Buchar.

(*Communicated by Prof. W. Heinrich.*)

1. *Eclipse totale de la Lune le 20 février* 1924.

On a observé les sorties du milieu des craters suivants :

T.M.Gr. h m	
5 10·0	Aristarch.
20·5	Cap Laplace.
22·5	Tycho.
25·5	Plato.
32·0	Aristillus.
36·0	Manilius.
38·5	Menelaus.
43·5	Plinius.
51·0	Mare Crisium, I bord, visible déjà dans l'ombre.
51·7	Proclus.
54·0	Mare Crisium, milieu.
56·0	Mare Crisium, II bord.
59·0	Fin de l'éclipse.

Le ciel a été beau, excepté quelques nuages au début. Le bord de l'ombre pure a été mal défini. L'instrument utilisé : 96 mm. Dollond, grossissement 58 ×.

2. *Eclipse partielle de la Lune le 8 février* 1925.

On n'a pu observer que la première partie du phénomène, les nuages ayant empêché l'observation complète. On a obtenu les entrées que voici :

T.U. h m s	
20 8 26	Commencement, incertain.
14 16	Schikard, I bord.
15 56	Grimaldi, I bord.
17 4	Grimaldi, milieu.
18 31	Grimaldi, II bord.
25 50	Campanus.
27 51	Reiner.
30 18	Tycho, I.
31 46	Tycho, II.
33 32	Pitatus.
37 53	Kepler.
41 41	Seleucus.
43 36	Fabricius.
47 6	Milichius.
48 51	Kopernicus, I.
50 54	Kopernicus, milieu.
52 16	Kopernicus, II.
20 53 26	Aristarch.
54 56	Herodot, nuages.
21 2 46	Furnerius.
14 16	Manilius.
15 40	Censorinus.
18 58	Langrenus, I.
22 6	Langrenus, milieu.
23 56	Langrenus, II.
23 56	Menelaus.
26 16	Plinius.
32 16	Archimedes, I bord.
37 56	Mare Crisium, I bord.
40 33	Proclus.
45 26	Herodot, sortie.
47 18	Aristarch, sortie.
48 36	Mare Crisium, milieu.
22 1 56	Mare Crisium, II bord, le ciel se couvre.

On a utilisé une petite lunette (objectif 74 mm., gross. 56 ×). Nuages par instants. La partie éclipsée de la lune était de couleur rouge cuivré, verdâtre vers la limite de l'ombre.

3. *Occultations d'Aldebaran par la Lune.*

Date.	Phén.	T.U. h m s	Gross.	Remarques.
1924 16 octobre	I	7 27 42·4*	190	Etoile se perd dans les ondulations du bord lunaire.
	E	8 21 19·1	62	Instantanée.
1925 6 janvier	E	16 4 50·6	96	Nuages.
1925 3 février	I	0 1 32·0	62	A travers des nuages.

L'œil et oreille.

4. *Occultation de la planète Mars par la Lune le 5 novembre 1925.*

On a pu enregistrer les contacts du bord avec le disque de Mars, présentant une phase :

	Phén.	T.M.Gr. h m s	Remarques.
II contact	I	8 4 13·1	
III „	E	9 1 24	Trop tard ; il faut retrancher à peu près 4ˢ.
IV „	E	9 1 57·9	

Grossissement utilisé, 190 ×. Une légère couche de cirrus au commencement. La luminosité du disque était un peu plus forte que celle des parties les plus foncées de la Lune. Toutes ces occultations ont été observées avec le réfracteur de 217 mm.

Nous pouvons ajouter les coordonnées de l'Observatoire que voici :

$$\lambda = 0^h\ 57^m\ 35^s \text{ à l'est de Greenwich.}$$
$$\phi = 50^\circ\ 4'\ 35'' \text{ N.}$$

altitude au dessus du niveau de mer étant 270 m.

Radiative Equilibrium in Inner Layers of Stars.

By V. A. Ambarzumian and N. A. Kosirev.

(*Communicated by the Secretaries.*)

§ 1. In this paper we try to prove that Neuman's series for the resolvent of the kernel $Ei\,|\,\tau - t\,|$, which plays the essential part in the question of radiative equilibrium, converges, the value of the parameter being $\lambda = \frac{1}{2}$; and that it represents an integrable function, in every finite interval at least. Thus we obtain the solution of the integral equation

$$f(\tau) = B(\tau) - \tfrac{1}{2}\int_0^\infty Ei\,|\,\tau - t\,|\,B(t)dt \quad . \quad . \quad (1)$$

---

* T.M.Gr.

where B(t) is the radiation of an absolutely black body at the temperature of the given layer, the optical mass therein being equal to t, $f(\tau) = \frac{\epsilon}{k}$, $4\pi\epsilon$ the quantity of energy formed per unit mass, k the coefficient of the absorption, in the form

$$B(\tau) = f(\tau) + \tfrac{1}{2}\int_0^\infty \Gamma(\tau, t) f(t) dt \quad . \quad . \quad . \quad (2)$$

where

$$\Gamma(\tau, t) = Ei \mid \tau - t \mid + \tfrac{1}{2}k^{(2)}(\tau, t) + \tfrac{1}{4}k^{(3)}(\tau, t) + \ldots \quad . \quad (3)$$

$\Gamma(\tau, t)$ represents the resolvent, $k^{(n)}(\tau, t)$ — nth-iterated kernel obtained from $Ei \mid \tau - t \mid$, as all the iterated kernels are positive and therefore (3) admits an integration term by term. The function $\Gamma(\tau, t)$ is calculated once for all and is independent of the form of $f(\tau)$. Consequently the solution of the equation of radiative equilibrium (1) by means of formula (2) has many advantages compared with other methods.

§ 2. Let us denote

$$\psi_1(\tau) = \tfrac{1}{2}(e^{-\tau} - \tau Ei\tau) = 1 - \tfrac{1}{2}\int_0^\infty Ei \mid \tau - t \mid dt \quad . \quad (4)$$

and introduce the function $\psi_n(\tau)$ by means of the recurrent relation

$$\psi_n(\tau) = \tfrac{1}{2}\int_0^\infty Ei \mid \tau - t \mid \psi_{n-1}(t) dt \quad . \quad . \quad . \quad (5)$$

We have already proved * that the series

$$\psi_1(\tau) + \psi_2(\tau) + \psi_3(\tau) + \ldots + \psi_n(\tau) + \ldots \quad . \quad (6)$$

converges and even uniformly in all cases when $\tau > 0$. We may write series (6) as follows :—

$$\psi_1(\tau) + \tfrac{1}{2}\int_0^\infty Ei \mid \tau - t \mid \psi_1(t) dt + \tfrac{1}{4}\int_0^\infty k^{(2)}(\tau, t)\psi_1(t) dt + \ldots + \frac{1}{2^n}\int_0^\infty k^{(n)}(\tau, t)\psi_1(t) dt + \ldots \quad (7)$$

Proceeding from the convergence of series (7) we shall prove the convergence of the series

$$Ei \mid \tau - t \mid \psi_1(t) + \tfrac{1}{2}k^{(2)}(\tau, t)\psi_1(t) + \ldots + \frac{1}{2^{n-1}}k^{(n)}(\tau, t)\psi_1(t) + \ldots \quad (8)$$

from which the convergence of (3) follows.

First of all we notice that series (8) is composed merely of positive terms. First let us prove that τ being constant, series (8) will be convergent for all $t > 0$ except perhaps the points of a set of measure zero (Lebesgue). Let us denote

$$S_n(\tau, t) = Ei \mid \tau - t \mid \psi_1(t) + \tfrac{1}{2}k^{(2)}(\tau, t)\psi_1(t) + \ldots + \frac{1}{2^{n-1}}k^{(n)}(\tau, t)\psi_1(t) \quad . \quad . \quad . \quad (9)$$

* *M.N.*, **87**, 213.

Since the sequence $S_n(\tau, t)$ is a monotonically increasing one, it may tend in every point either to a finite limit or to $+\infty$. At any rate in every point a certain finite or infinite limit does exist. Since all $S_n(\tau, t)$ in every finite interval represent measurable functions (all $k^{(n)}(\tau, t)$ being measurable functions), the limit $S_n(\tau, t)$ will also be a measurable function according to a well-known theorem. Consequently in every finite interval (a, b), $(a>0, b>0)$ the set of points in which this limit is $+\infty$ will have a certain measure, which we denote by m. We have now to show that $m = 0$. This is readily done if we assume the contrary, according to the following theorem the proof of which we omit for brevity. If the sequence of measurable functions $f_1, f_2, \ldots$ in a certain interval (a, b) is increasing monotonically, and if the set of points in which this sequence tends to $+\infty$ has the measure $m>0$, no matter what the positive number M and η may be, it is possible to find a number N such that in all cases $n>N$ the set of points, in which $f_n(\tau)>M$, has the measure $>m-\eta$.

The sequence of functions $S_n(\tau, t)$ satisfies the conditions of this theorem. Let us divide the set of points into two sets. In the first set E there are supposed to be all the points in which $S_n(\tau, t)\to+\infty$, in the second E′ all the remaining points. If we examine the integrals in the Lebesgue sense we get

$$\int_0^\infty S_n(\tau, t)dt = \int_E S_n(\tau, t)dt + \int_{E'} S_n(\tau, t)dt \quad . \quad . \quad (10)$$

By means of the above theorem we obtain

$$\int_0^\infty S_n(\tau, t)dt \geqslant M(m-\eta) + \int_{E'} S_n(\tau, t)dt \quad . \quad . \quad (11)$$

Since M can be made as great as we please by increasing n, it follows from (11) that, with increase of n, $\int_0^\infty S_n(\tau, t)dt$ tends to infinity. Now according to the notation for $S_n(\tau, t)$ we have

$$\int_0^\infty S_n(\tau, t)dt = \psi_2(\tau) + \psi_3(\tau) + \ldots + \psi_{n+1}(\tau) \quad . \quad (12)$$

But the expression (12) for any value of τ, as mentioned above, tends, and tends even uniformly, to a finite limit as n increases. We have thus come to a contradiction. Consequently $m = 0$.

§ 3. From (3) and (9) we obtain

$$\lim_{n\to\infty} S_n(\tau, t) = \Gamma(\tau, t)\psi_1(t) \quad . \quad . \quad . \quad (13)$$

Let us prove that in every finite interval $\Gamma(\tau, t)$ represents an integrable function. First of all let us show that $\Gamma(\tau, t)\psi_1(t)$ in every finite interval is an integrable function. In fact, we take any positive number M and divide the set of points of our finite interval into two sets in which $\Gamma(\tau, t)\psi_1(t)$ is less than or equal to M(E) and greater than $M(E_1)$. According to a well-known theorem we have for the first set

$$\lim_{n\to\infty} \int_E S_n(\tau, t)dt = \int_E \Gamma(\tau, t)\psi_1(t)dt \quad . \quad . \quad (14)$$

Now let M increase infinitely. If the integral $\int\Gamma(\tau, t)\psi_1(t)dt$ which is extended over the whole interval does not exist, the right-hand side of (14) tends to infinity. Meanwhile for all values of M, $\lim_{n\to\infty}\int_E S_n(\tau, t)dt$ is finite. Consequently $\int\Gamma(\tau, t)\psi_1(t)dt$ does exist, since we have come to a contradiction. Since $\psi_1(t)$ is a continuous function, and moreover a function which cannot be transformed into zero, we may conclude that $\Gamma(\tau, t)$ in every finite region $0 \leqslant a \leqslant t \leqslant b$ is integrable.

From the above we easily conclude that for every positive continuous function $f(t)$ we have

$$\int_0^A \Gamma(\tau, t)f(t)dt = \int_0^A \mathrm{E}i\,|\,\tau - t\,|\,f(t)dt + \tfrac{1}{2}\int_0^A k^{(2)}(\tau, t)f(t)dt + \ldots$$

From this we conclude that

$$\int_0^\infty \Gamma(\tau, t)f(t)dt = \int_0^\infty \mathrm{E}i\,|\,\tau - t\,|\,f(t)dt + \tfrac{1}{2}\int_0^\infty k^{(2)}(\tau, t)f(t)dt + \ldots \quad (15)$$

if the series of the right-hand side converges. The convergence of this series depends on the form of the function $f(t)$. For instance, this series converges for the particular case $f(t) = \psi_1(t)$. If series (15) converges the solution (2) for the equation (1) will be true, as one can easily see.

§ 4. Dr. Jeans in his very interesting paper "The Exact Equations of Radiative Equilibrium" * obtains the expressions for the intensity of transmitted energy I, for the density of energy R and for the radiative pressure p_R, by means of the following infinite differential operations

$$\mathrm{I} = \frac{\mathrm{B}}{4\pi k} + \cos\theta\frac{\partial}{\partial\tau}\left(\frac{\mathrm{B}}{4\pi k}\right) + \left(\cos^2\theta + \cos^3\theta\frac{\partial}{\partial\tau} + \cos^4\theta\frac{\partial^2}{\partial\tau^2} + \ldots\right)\left(-3 + \frac{9}{5}\frac{\partial^2}{\partial\tau^2} + \ldots\right)\left(\frac{4\pi\epsilon}{k}\right) \quad (16)$$

$$\mathrm{R} = \frac{\mathrm{B}}{\mathrm{C}k} + \left(\frac{1}{3\mathrm{C}} + \frac{1}{5\mathrm{C}}\frac{\partial^2}{\partial\tau^2} + \frac{1}{7\mathrm{C}}\frac{\partial^4}{\partial\tau^4} + \ldots\right)\left(-3 + \frac{9}{5}\frac{\partial^2}{\partial\tau^2} + \ldots\right)\left(\frac{4\pi\epsilon}{k}\right) \quad (17)$$

$$p_\mathrm{R} = \frac{\mathrm{B}}{3\mathrm{C}k} + \left(\frac{1}{5\mathrm{C}} + \frac{1}{7\mathrm{C}}\frac{\partial^2}{\partial\tau^2} + \frac{1}{9\mathrm{C}}\frac{\partial^4}{\partial\tau^4} + \ldots\right)\left(-3 + \frac{9}{5}\frac{\partial^2}{\partial\tau^2} + \ldots\right)\left(\frac{4\pi\epsilon}{k}\right) \quad (18)$$

By means of the relations we have obtained above we can give to these formulæ a more simple form.

First of all we obtain for I

$$\mathrm{I} = \mathrm{B} + \cos\theta\frac{d\mathrm{B}}{d\tau} + 4\pi\left(\cos^2\theta + \cos^3\theta\frac{d}{d\tau} + \cos^4\theta\frac{d}{d\tau} + \ldots\right)\left(f(\tau) + \tfrac{1}{2}\int_0^\infty \Gamma(\tau, t)f(t)dt\right) \quad (19)$$

* *M.N.*, **86**, 574, 1926.

Proceeding from the condition of radiative equilibrium

$$B(\tau) = f(\tau) + \frac{C}{4\pi} R(\tau) \quad . \quad . \quad . \quad (20)$$

where C is the velocity of light. By means of formula (2) we obtain

$$R(\tau) = 2\pi \int_0^\infty \Gamma(\tau, t) f(t) dt \quad . \quad . \quad . \quad (21)$$

And finally we get for p_R the following expression :—

$$p_R = \frac{4\pi R}{3C} + 4\pi \left(\frac{1}{5C} + \frac{1}{7C} \frac{d^2}{d\tau^2} + \frac{1}{9C} \frac{d^4}{d\tau^4} + \dots \right) \left(f(\tau) + \tfrac{1}{2} \int_0^\infty \Gamma(\tau, t) f(t) dt \right) \quad (22)$$

As for the flow of radiative energy through a unit of the surface we have the following simple expression :—

$$H(\tau) = - \int_0^\tau f(t) dt + C_1 \quad . \quad . \quad . \quad (23)$$

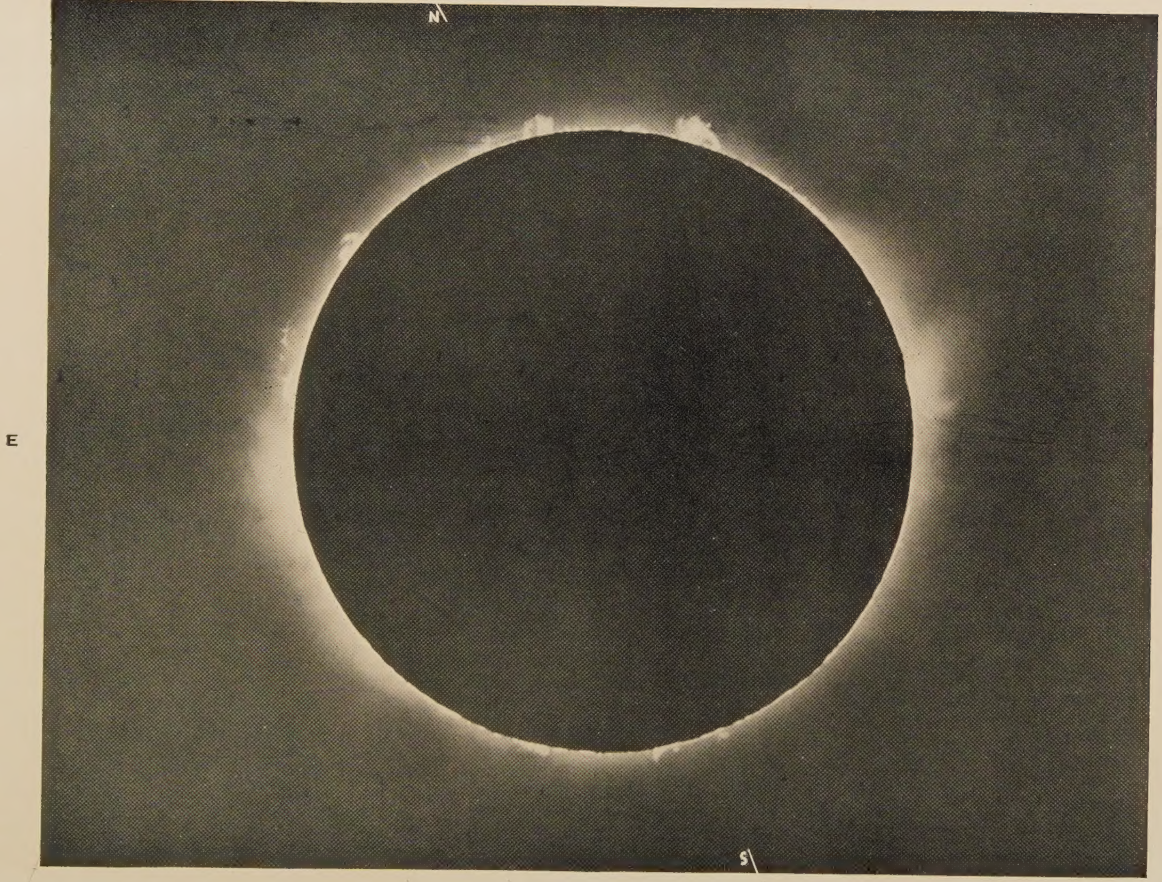
where C_1 is a constant.

§ 5. From the formulæ (19) and (22) there follows the correctness of Jeans' statement that the error in Eddington's formula attains for I the order $f(\tau)$. A similar error (Eddington, Zeipel) occurs in the generally assumed formula for the radiative pressure p_R.

$$p_R = \tfrac{1}{3} \alpha T^4 \quad . \quad . \quad . \quad . \quad (24)$$

where T is the temperature and α a constant. Thus any theorem based on formula (24) and giving the expression $f(\tau)$ is deprived of its basis.

Leningrad :
1927 *May* 15.



Solar Corona photographed at Giggleswick. Exp. 19^{s}. Royal Observatory, Greenwich.
The letters N and S refer to the Sun's axis.